CMSC 330: Organization of Programming Languages

Operational Semantics

Formal Semantics of a Prog. Lang.

- Mathematical description of the meaning of programs written in that language
  - What a program computes, and what it does

`"1+2"` `Plus(Int 1, Int 2)`

Parse

Concrete Syntax

Abstract Syntax

- What does `Plus(Int 1, Int 2)` mean?

Operational semantics

- Define how programs execute
  - Often on an abstract machine (mathematical model of computer)
  - Analogous to interpretation

- We will define an operational semantics for Micro-Ocaml
  - And develop an interpreter for it, along the way

- Approach: use rules to define a judgment

$$e \Rightarrow v$$

Micro-OCaml Expression Grammar

e ::= x | n | e + e | `let` x `=` $e1$ `in` $e2$

Corresponding AST:

```
type id = string
type exp =
    | Ident of id                (* x *)
    | Num of int                 (* n *)
    | Plus of exp * exp          (* e+e *)
    | Let of id * exp * exp (* let x=e1 in e2 *)
```

Defining the Semantics

- Use rules to define judgment $e \Rightarrow v$
- Inference Rules

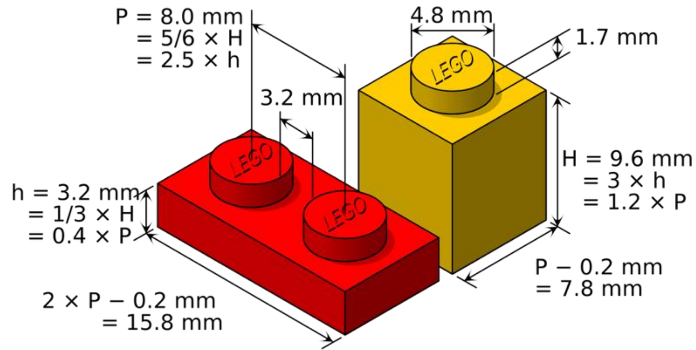
$$\frac{H_1, \; H_2 \; \ldots \; H_n}{C}$$

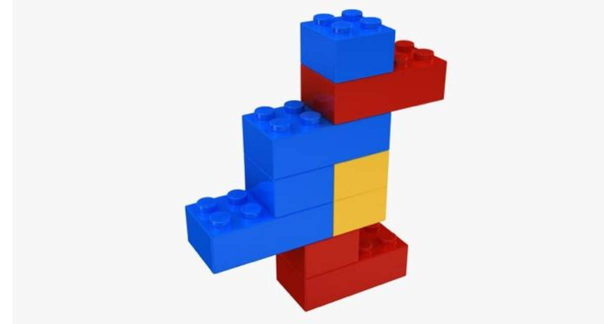
$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$\frac{Man(Socrates)}{\therefore \quad Mortal(Socrates)}$$

$$H_1 \wedge H_2 \wedge \ldots H_n \Rightarrow C$$

Rules are Lego Blocks





Rules of Inference: Num and Sum

$$\frac{}{n \Rightarrow n}$$

axiom

```
match e with
| Num n -> n
| Plus (e1,e2) ->
     let n1 = eval e1 in
     let n2 = eval e2 in
     let n3 = n1 + n2 in
     n3
```

$$\frac{e1 \Rightarrow n1 \qquad e2 \Rightarrow n2 \qquad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

Rules of Inference: Let

$$\frac{e1 \Rightarrow v1 \qquad e2\{v1/x\} \Rightarrow v2}{\texttt{let}\ x\ \texttt{=}\ e1\ \texttt{in}\ e2 \Rightarrow v2}$$

```
match e with
| Let (x,e1,e2) ->
      let v1 = eval e1 in
      let e2' = subst v1 x e2 in
      let v2 = eval e2'
   in v2
```

Derivations

- When we apply rules to an expression in succession, we produce a derivation
  - It's a kind of tree, rooted at the conclusion

- Produce a derivation by goal-directed search
  - Pick a rule that could prove the goal
  - Then repeatedly apply rules on the corresponding hypotheses
    - Goal: Show that `let x = 4 in x+3` ⇒ `7`

Derivations

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \quad e2 \Rightarrow n2 \quad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Goal: show that

`let x = 4 in x+3` ⇒ `7`

$$\frac{4 \Rightarrow 4 \qquad \dfrac{4 \Rightarrow 4 \quad 3 \Rightarrow 3 \quad 7 \text{ is } 4+3}{4+3 \Rightarrow 7}}{\texttt{let x = 4 in x+3} \Rightarrow 7}$$

Definitional Interpreter

- The style of rules lends itself directly to the implementation of an interpreter as a recursive function

```
let rec eval (e:exp):value =
  match e with
    Ident x -> (* no rule *)
     failwith “no value”
  | Num n -> n
  | Plus (e1,e2) ->
     let n1 = eval e1 in
     let n2 = eval e2 in
     let n3 = n1+n2 in
     n3
  | Let (x,e1,e2) ->
     let v1 = eval e1 in
     let e2’ = subst v1 x e2 in
     let v2 = eval e2’ in v2
```

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \qquad e2 \Rightarrow n2 \qquad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \qquad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Derivations = Interpreter Call Trees

$$\frac{4 \Rightarrow 4 \qquad \dfrac{3 \Rightarrow 3 \qquad 7 \text{ is } 4+3}{4+3 \Rightarrow 7}}{\texttt{let x = 4 in x+3} \Rightarrow 7}$$

Has the same shape as the recursive call tree of the interpreter:

$$\frac{\texttt{eval Num 4} \Rightarrow 4 \qquad \dfrac{\texttt{eval Num 4} \Rightarrow 4 \qquad \texttt{eval Num 3} \Rightarrow 3 \qquad 7 \text{ is } 4+3}{\texttt{eval (subst 4 "x" Plus(Ident("x"),Num 3))} \Rightarrow 7}}{\texttt{eval Let("x",Num 4,Plus(Ident("x"),Num 3))} \Rightarrow 7}$$

Semantics Defines Program Meaning

- $e \Rightarrow v$ holds if and only if a *proof* can be built
  - Proofs are derivations: axioms at the top, then rules whose hypotheses have been proved to the bottom
  - No proof means there *exists no* v *for which* $e \Rightarrow v$
- Proofs can be constructed bottom-up
  - In a goal-directed fashion
- Thus, function eval $e = \{v \mid e \Rightarrow v\}$
  - Determinism of semantics implies at most one element for any e
- So: Expression e *means* v

Environment-style Semantics

- So far, semantics used substitution to handle variables
  - As we evaluate, we replace all occurrences of a variable **x** with values it is bound to

- An alternative semantics, closer to a real implementation, is to use an environment
  - As we evaluate, we maintain an explicit map from variables to values, and look up variables as we see them

Environments

- Mathematically, an environment is a partial function from identifiers to values
  - If A is an environment, and ***x*** is an identifier, then A(***x***) can either be
    - a value ***v*** (intuition: the value of the variable stored on the stack)
    - undefined (intuition: the variable has not been declared)
- An environment can visualized as a table
  - If A is

Id	Val
`x`	0
`y`	2

  - then A(`x`) is `0`, A(`y`) is `2`, and A(`z`) is undefined

Notation, Operations on Environments

- • is the empty environment
- A,x:v is the environment that extends A with a mapping from x to v
  - Sometimes just write x:v instead of •,x:v for brevity
- Lookup A(x) is defined as follows

$$\bullet(x) = \text{undefined}$$

$$(A, y{:}v)(x) = \begin{cases} v & \text{if } x = y \\ A(x) & \text{if } x <> y \text{ and } A(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Definitional Interpreter: Environments

```
type env = (id * value) list

let extend env x v = (x,v)::env

let rec lookup env x =
  match env with
    [] -> failwith "undefined"
  | (y,v)::env' ->
    if x = y then v
    else lookup env' x
```

An environment is just a list of mappings, which are just pairs of variable to value
- called an association list

Semantics with Environments

- The environment semantics changes the judgment

$$e \Rightarrow v$$

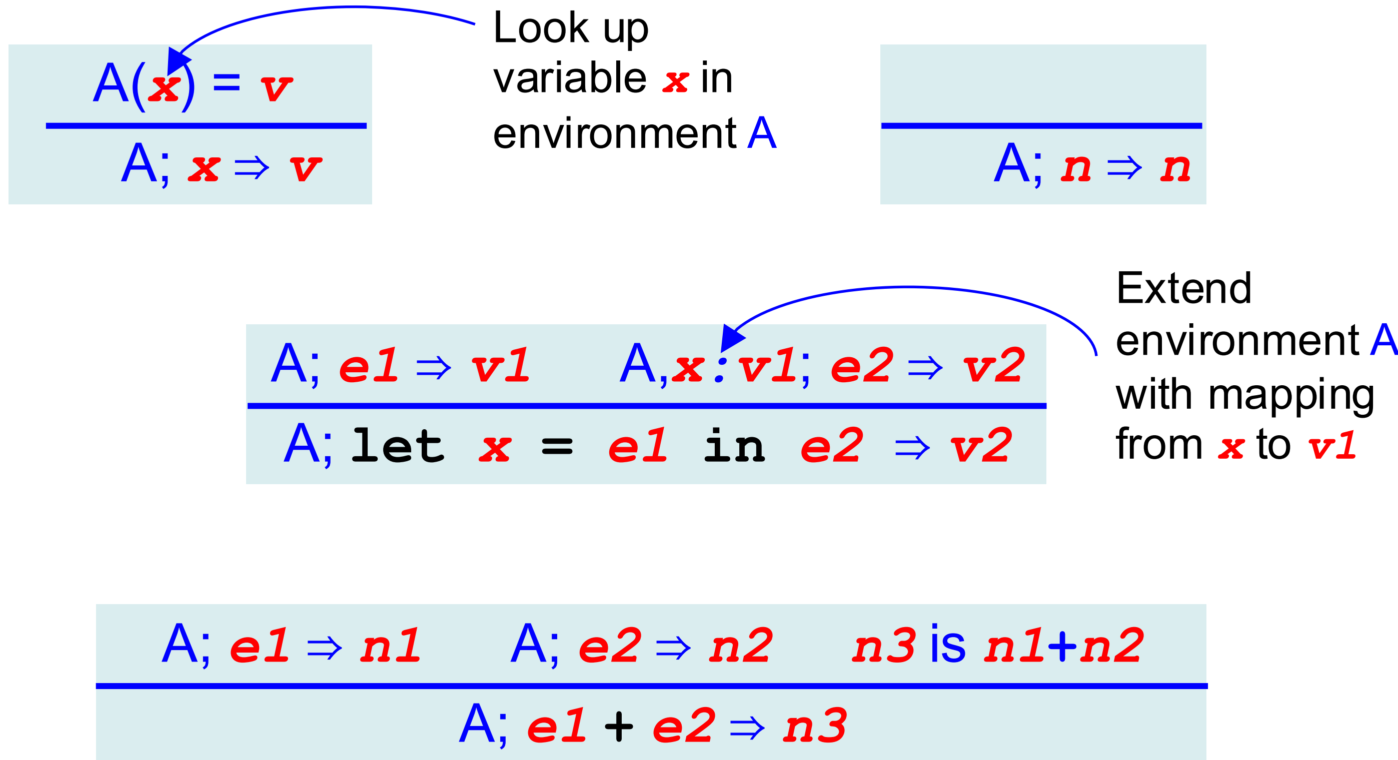
to be

$$A;\ e \Rightarrow v$$

where A is an environment

  - Idea: A is used to give values to the identifiers in e

Environment-style Rules



Definitional Interpreter: Evaluation

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Num n -> n
  | Plus (e1,e2) ->
     let n1 = eval env e1 in
     let n2 = eval env e2 in
     let n3 = n1+n2 in
     n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
```

Quiz 1

Describe in English what the operator myst does

$$\text{Mystery}(1): \frac{A;\ e_1 \Rightarrow true \quad A;\ e_2 \Rightarrow true}{A;\ e_1\ myst\ e_2 \Rightarrow false}$$

$$\text{Mystery}(2): \frac{A;\ e_1 \Rightarrow false \quad A;\ e_2 \Rightarrow false}{A;\ e_1\ myst\ e_2 \Rightarrow false}$$

$$\text{Mystery}(3): \frac{A;\ e_1 \Rightarrow true \quad A;\ e_2 \Rightarrow false}{A;\ e_1\ myst\ e_2 \Rightarrow true}$$

$$\text{Mystery}(4): \frac{A;\ e_1 \Rightarrow false \quad A;\ e_2 \Rightarrow true}{A;\ e_1\ myst\ e_2 \Rightarrow true}$$

Quiz 1

Describe in English what the operator myst does

XOR

$$\text{Mystery}(1): \frac{A;\ e_1 \Rightarrow true \qquad A;\ e_2 \Rightarrow true}{A;\ e_1\ myst\ e_2 \Rightarrow false}$$

$$\text{Mystery}(2): \frac{A;\ e_1 \Rightarrow false \qquad A;\ e_2 \Rightarrow false}{A;\ e_1\ myst\ e_2 \Rightarrow false}$$

$$\text{Mystery}(3): \frac{A;\ e_1 \Rightarrow true \qquad A;\ e_2 \Rightarrow false}{A;\ e_1\ myst\ e_2 \Rightarrow true}$$

$$\text{Mystery}(4): \frac{A;\ e_1 \Rightarrow false \qquad A;\ e_2 \Rightarrow true}{A;\ e_1\ myst\ e_2 \Rightarrow true}$$

Describe what the operator myst does

$$\frac{A; e_1 \rightarrow true \qquad A; e_2 \rightarrow true}{A; e_1 \ \mathbf{myst} \ e_2 \rightarrow true}$$

$$\frac{A; e_1 \rightarrow true \qquad A; e_2 \rightarrow false}{A; e_1 \ \mathbf{myst} \ e_2 \rightarrow false}$$

$$\frac{A; e_1 \rightarrow false \qquad A; e_2 \rightarrow true}{A; e_1 \ \mathbf{myst} \ e_2 \rightarrow false}$$

$$\frac{A; e_1 \rightarrow false \qquad A; e_2 \rightarrow false}{A; e_1 \ \mathbf{myst} \ e_2 \rightarrow false}$$

Describe what the operator myst does: AND

$$\frac{A; e_1 \rightarrow true \qquad A; e_2 \rightarrow true}{A; e_1 \textbf{ myst } e_2 \rightarrow true}$$

$$\frac{A; e_1 \rightarrow true \qquad A; e_2 \rightarrow false}{A; e_1 \textbf{ myst } e_2 \rightarrow false}$$

$$\frac{A; e_1 \rightarrow false \qquad A; e_2 \rightarrow true}{A; e_1 \textbf{ myst } e_2 \rightarrow false}$$

$$\frac{A; e_1 \rightarrow false \qquad A; e_2 \rightarrow false}{A; e_1 \textbf{ myst } e_2 \rightarrow false}$$

Adding Conditionals to Micro-OCaml

e ::= ***x*** | ***v*** | ***e*** **+** ***e*** | **let** ***x*** **=** ***e*** **in** ***e***
| **eq0** ***e*** | **if** ***e*** **then** ***e*** **else** ***e***

v ::= ***n*** | **true** | **false**

- In terms of interpreter definitions:

```
type exp =
  | Val of value
  | … (* as before *)
  | Eq0 of exp
  | If of exp * exp * exp
```

```
type value =
    Int of int
  | Bool of bool
```

Rules for Eq0 and Booleans

$$\frac{}{A;\ \texttt{true} \Rightarrow \texttt{true}}$$

$$\frac{A;\ e \Rightarrow 0}{A;\ \texttt{eq0}\ e \Rightarrow \texttt{true}}$$

$$\frac{}{A;\ \texttt{false} \Rightarrow \texttt{false}}$$

$$\frac{A;\ e \Rightarrow v \qquad v \neq 0}{A;\ \texttt{eq0}\ e \Rightarrow \texttt{false}}$$

Rules for Conditionals

$$\frac{A;\ e1 \Rightarrow \texttt{true} \qquad A;\ e2 \Rightarrow v}{A;\ \texttt{if}\ e1\ \texttt{then}\ e2\ \texttt{else}\ e3 \Rightarrow v}$$

$$\frac{A;\ e1 \Rightarrow \texttt{false} \qquad A;\ e3 \Rightarrow v}{A;\ \texttt{if}\ e1\ \texttt{then}\ e2\ \texttt{else}\ e3 \Rightarrow v}$$

- Notice that only one branch is evaluated

Updating the Interpreter

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Val v -> v
  | Plus (e1,e2) ->
     let Int n1 = eval env e1 in
     let Int n2 = eval env e2 in
     let n3 = n1+n2 in
     Int n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
  | Eq0 e1 ->
     let Int n = eval env e1 in
     if n=0 then Bool true else Bool false
  | If (e1,e2,e3) ->
     let Bool b = eval env e1 in
     if b then eval env e2
     else eval env e3
```

Adding Closures to Micro-OCaml

e ::= x | v | e + e | let x = e in e
| eq0 e | if e then e else e
| e e | fun x -> e

v ::= n | true | false | (A, λx.e)

Environment

Code
(id and exp)

- In terms of interpreter definitions:

```
type exp =
  | Val of value
  | If of exp * exp * exp
   … (* as before *)
  | Call of exp * exp
  | Fun of id * exp
```

```
type value =
    Int of int
  | Bool of bool
  | Closure of env * id * exp
```

Rule for Closures: Lexical/Static Scoping

$$\frac{}{A;\ \texttt{fun}\ x\ \texttt{->}\ e \Rightarrow (A, \lambda x.e)}$$

$$\frac{A;\ e1 \Rightarrow (A', \lambda x.e) \qquad A;\ e2 \Rightarrow v1 \qquad A', x:v1;\ e \Rightarrow v}{A;\ e1\ e2 \Rightarrow v}$$

- Notice
  - Creating a closure captures the current environment A
  - A call to a function
    - evaluates the body of the closure's code e with function closure's environment A' extended with parameter x bound to argument $v1$

Rule for Closures: Dynamic Scoping

$$\frac{}{A;\ \texttt{fun}\ x\ \texttt{->}\ e \Rightarrow (\bullet, \lambda x.e)}$$

$$\frac{A;\ e1 \Rightarrow (\bullet, \lambda x.e) \qquad A;\ e2 \Rightarrow v1 \qquad A, x:v1;\ e \Rightarrow v}{A;\ e1\ e2 \Rightarrow v}$$

- Notice
  - Creating a closure ignores the current environment A
  - A call to a function
    - evaluates the body of the closure's code e with the current environment A extended with parameter x bound to argument $v1$

Scaling up

- Operational semantics can handle full languages
  - With records, recursive variant types, objects, first-class functions, and more

- Provides a concise notation for explaining what a language does. Clearly shows:
  - Evaluation order
  - Call-by-value vs. call-by-name
  - Static scoping vs. dynamic scoping
  - ... We may look at more of these later

Scaling up: Lego City