CMSC 330: Organization of Programming Languages

Regular Expressions and Finite Automata

How do regular expressions work?

- What we've learned
  - What regular expressions are
  - What they can express, and cannot
  - Programming with them

- What's next: how they work
  - A great computer science result

A Few Questions About REs

- How are REs implemented?
  - Given an arbitrary RE and a string, how to decide whether the RE matches the string?
- What are the basic components of REs?
  - Can implement some features in terms of others
    - E.g., e+ is the same as ee*
- What does a regular expression represent?
  - Just a set of strings
    - This observation provides insight on how we go about our implementation

Definition: Alphabet

- An alphabet is a finite set of symbols
  - Usually denoted Σ
- Example alphabets:
  - Binary: $\Sigma = \{0,1\}$
  - Decimal: $\Sigma = \{0,1,2,3,4,5,6,7,8,9\}$
  - Alphanumeric: $\Sigma = \{0\text{-}9,a\text{-}z,A\text{-}Z\}$

Definition: String

- A string is a finite sequence of symbols from Σ
  - ε is the empty string ("" in OCaml)
  - $|s|$ is the length of string s
    - $|\text{Hello}| = 5$, $|\varepsilon| = 0$
  - Note
    - $\emptyset$ is the empty set (with 0 elements)
    - $\emptyset \neq \{ \varepsilon \}$ (and $\emptyset \neq \varepsilon$)
- Example strings over alphabet $\Sigma = \{0,1\}$ (binary):
  - 0101
  - 0101110
  - ε

Definition: Language

- A language L is a set of strings over an alphabet
- Example: All strings of length 1 or 2 over alphabet Σ = {a, b, c} that begin with a
  - L = { a, aa, ab, ac }
- Example: All strings over Σ = {a, b}
  - L = { ε, a, b, aa, bb, ab, ba, aaa, bba, aba, baa, … }
  - Language of all strings written Σ*

Definition: Language (cont.)

- Example: The set of phone numbers over the alphabet Σ = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, (,), -}
  - Give an example element of this language `(123)456-7890`
  - Are all strings over the alphabet in the language? **No**
  - Is there a regular expression for this language?

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\(\d{3}\)\d{3}-\d{4}
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- Example: The set of all valid (runnable) OCaml programs
  - Later we'll see how we can specify this language
  - (Regular expressions are useful, but not sufficient)

Operations on Languages

- Let Σ be an alphabet and let L, L_1, L_2 be languages over Σ
- Concatenation L_1L_2 creates a language defined as
  - $L_1L_2 = \{ xy \mid x \in L_1 \text{ and } y \in L_2\}$
- Union creates a language defined as
  - $L_1 \cup L_2 = \{ x \mid x \in L_1 \text{ or } x \in L_2\}$
- Kleene closure creates a language is defined as
  - $L^* = \{ x \mid x = \varepsilon \text{ or } x \in L \text{ or } x \in LL \text{ or } x \in LLL \text{ or } \ldots\}$

Operations Examples

Let L_1 = { a, b }, L_2 = { 1, 2, 3 } (and Σ = {a,b,1,2,3})

- What is L_1L_2 ?
  - { a1, a2, a3, b1, b2, b3 }
- What is $L_1 \cup L_2$?
  - { a, b, 1, 2, 3 }
- What is L_1* ?
  - { ε, a, b, aa, bb, ab, ba, aaa, aab, bba, bbb, aba, abb, baa, bab, … }

Quiz 1: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a,b,c,d\}$

$L_2 = \{d\}$

$L_3 = L_1 \cup L_2$

A. cd
B. c
C. ε
D. d

Quiz 1: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a,b,c,d\}$

$L_2 = \{d\}$

$L_3 = L_1 \cup L_2$

A. cd
B. c
C. ε
D. d

Quiz 2: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a,b,c,d\}$

$L_2 = \{d\}$

$L_3 = L_1(L_2{}^*)$

A. a
B. abd
C. abdd
D. adad

Quiz 2: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a,b,c,d\}$

$L_2 = \{d\}$

$L_3 = L_1(L_2{}^*)$

A. a
B. abd
C. abdd
D. adad

Regular Expressions: Grammar

- We can define a grammar for regular expressions R

$R ::= \varnothing$	The empty language
$\mid \varepsilon$	The empty string
$\mid \sigma$	A symbol from alphabet Σ
$\mid R_1R_2$	The concatenation of two regexps
$\mid R_1 \mathbin{\vert} R_2$	The union of two regexps
$\mid R^*$	The Kleene closure of a regexp

Regular Languages

- Regular expressions denote regular languages
- Not all languages are regular
  - Examples (without proof):
    - The set of palindromes over Σ
    - $\{a^n b^n \mid n > 0\}$ (a^n = sequence of *n* a's)
- Almost all programming languages are not regular
  - But aspects of them sometimes are (e.g., identifiers)
  - Regular expressions are commonly used in parsing tools

Semantics: Regular Expressions (1)

- Given an alphabet Σ, the regular expressions over Σ are defined inductively as follows

Constants

regular expression	denotes language
∅	∅
ε	{ε}
each symbol $\sigma \in \Sigma$	$\{\sigma\}$

Ex: with Σ = { a, b }, regex a denotes language {a}
regex b denotes language {b}

Semantics: Regular Expressions (2)

- Let A and B be regular expressions denoting languages L_A and L_B, respectively. Then:

Operations

regular expression	denotes language
AB	$L_A L_B$
A|B	$L_A \cup L_B$
A*	L_A^*

- There are no other regular expressions over Σ

Terminology etc.

- Regexps apply operations to symbols
  - Generates a set of strings (i.e., a language)
    - (Formal definition shortly)
  - Examples
    - a generates language {a}
    - a|b generates language {a} ∪ {b} = {a, b}
    - a* generates language {ε} ∪ {a} ∪ {aa} ∪ … = {ε, a, aa, … }

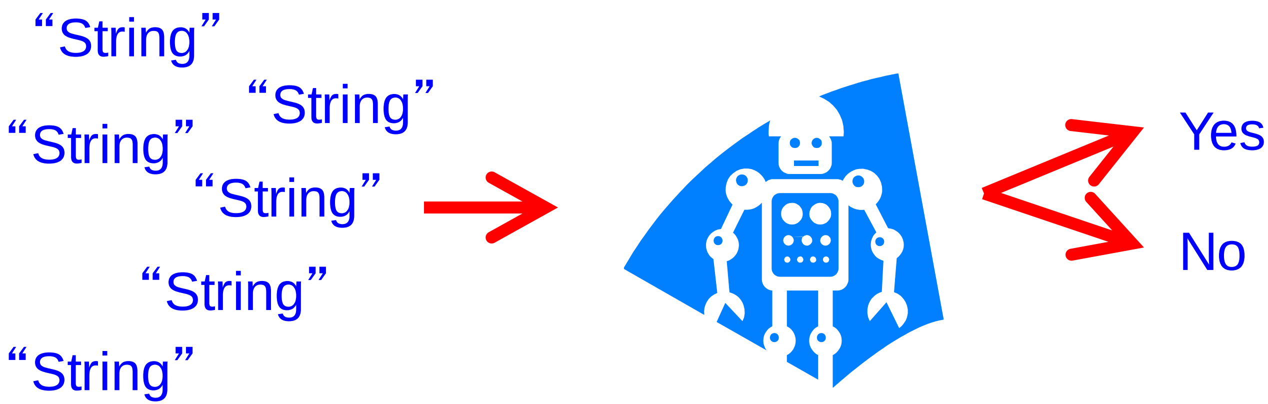
- If s ∊ language L generated by a RE r, we say that r accepts, describes, or recognizes string s

Regular Expressions

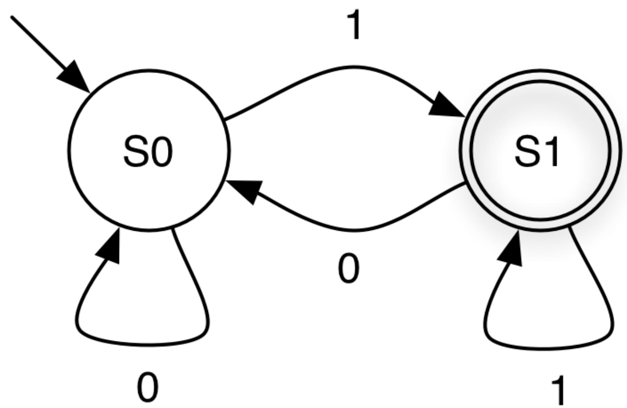
- Almost all of the features we've seen for REs can be reduced to this formal definition
  - OCaml – concatenation of single-symbol REs
  - /(OCaml|Rust)/ – union
  - /(OCaml)*/ – Kleene closure
  - /(OCaml)+/ – same as (OCaml)(OCaml)*
  - /(Ocaml)?/ – same as (ε|(OCaml))
  - /[a-z]/ – same as (a|b|c|...|z)
  - / [^0-9]/ – same as (a|b|c|...) for a,b,c,... ∈ Σ - {0..9}
  - ^, $ – correspond to extra symbols in alphabet
    - Think of every string containing a distinct, hidden symbol at its start and at its end – these are written ^ and $

Implementing Regular Expressions

- We can implement a regular expression by turning it into a finite automaton
  - A “machine” for recognizing a regular language



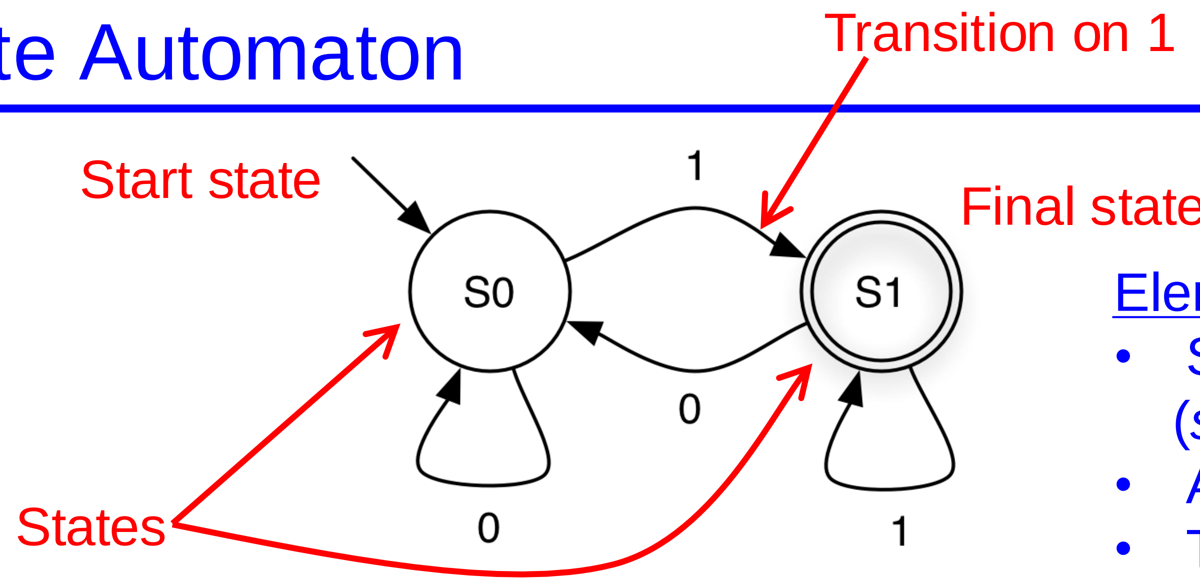
Finite Automaton



<u>Elements</u>

- States S (*start*, *final*)
- Alphabet Σ
- Transition edges δ

Finite Automaton



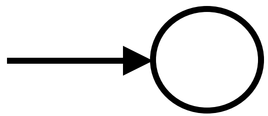
Elements

- States S
  (*start*, *final*)
- Alphabet Σ
- Transition edges δ

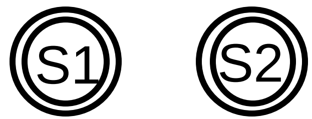
- Machine starts in start or initial state
- Repeat until the end of the string s is reached
  - Scan the next symbol $\sigma \in \Sigma$ of the string s
  - Take transition edge labeled with σ
- String s is accepted if automaton is in final state when end of string s is reached

Finite Automaton: States

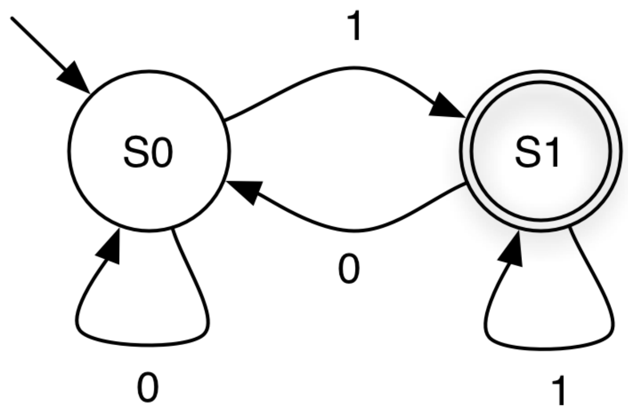
- Start state
  - State with incoming transition from no other state
  - Can have only one start state



- Final states
  - States with double circle
  - Can have zero or more final states
  - Any state, including the start state, can be final



Finite Automaton: Example 1

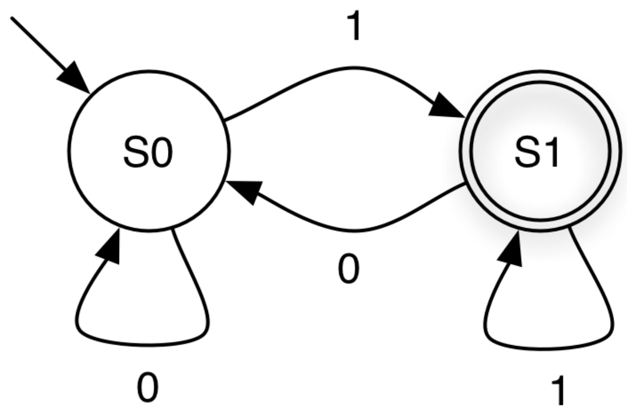


0 0 1 0 1 1

Accepted?

Yes

Finite Automaton: Example 2

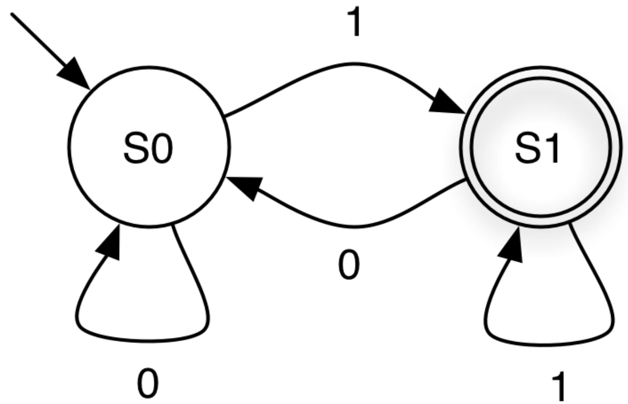


0 0 1 0 1 0

Accepted?

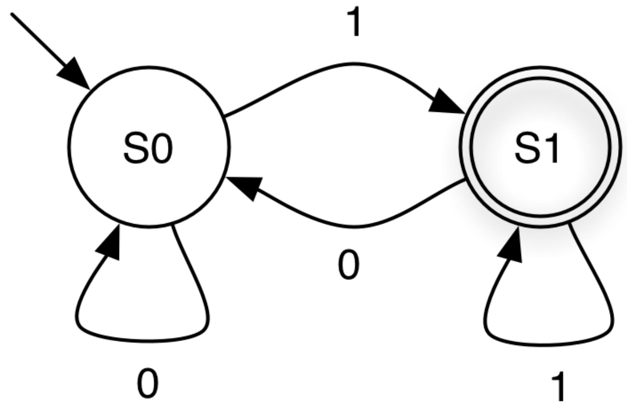
No

Quiz 3: What Language is This?



A. All strings over {0, 1}
B. All strings over {1}
C. All strings over {0, 1} of length 1
D. All strings over {0, 1} that end in 1

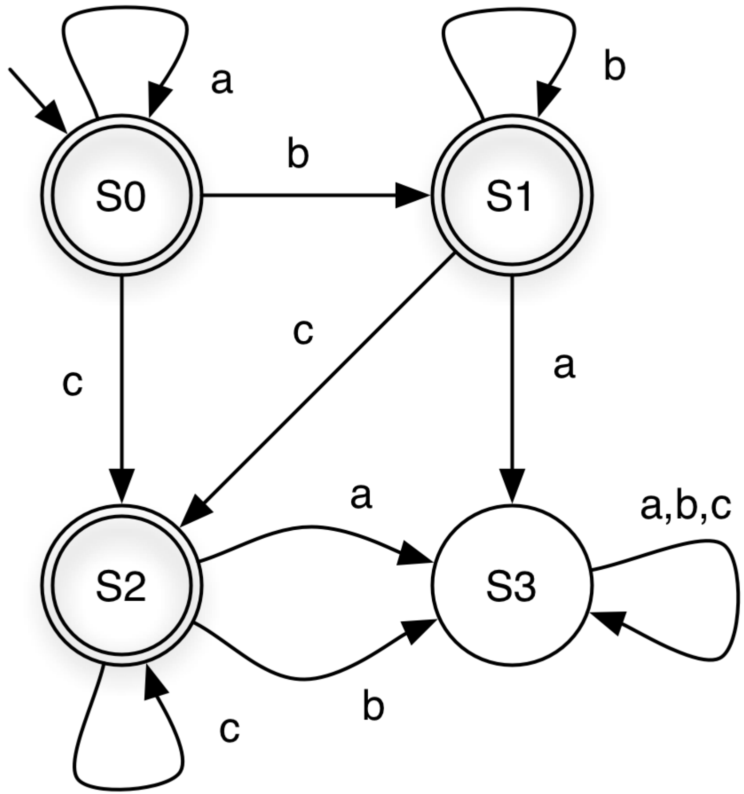
Quiz 3: What Language is This?



A. All strings over {0, 1}
B. All strings over {1}
C. All strings over {0, 1} of length 1
D. All strings over {0, 1} that end in 1

regular expression for this language is (0|1)*1

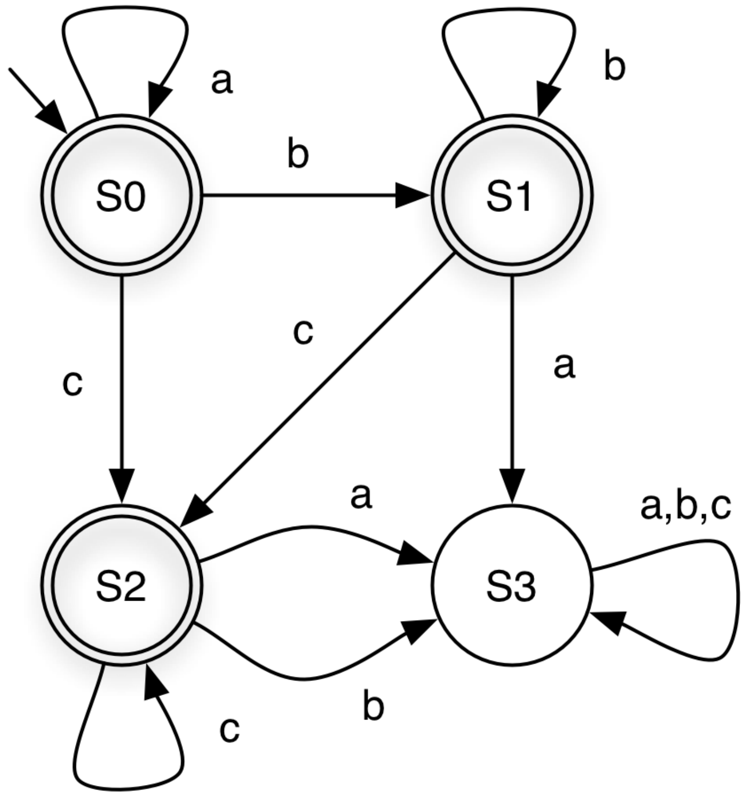
Finite Automaton: Example 3



string	state at end	accept s?
aabcc		

(a,b,c notation shorthand for three self loops)

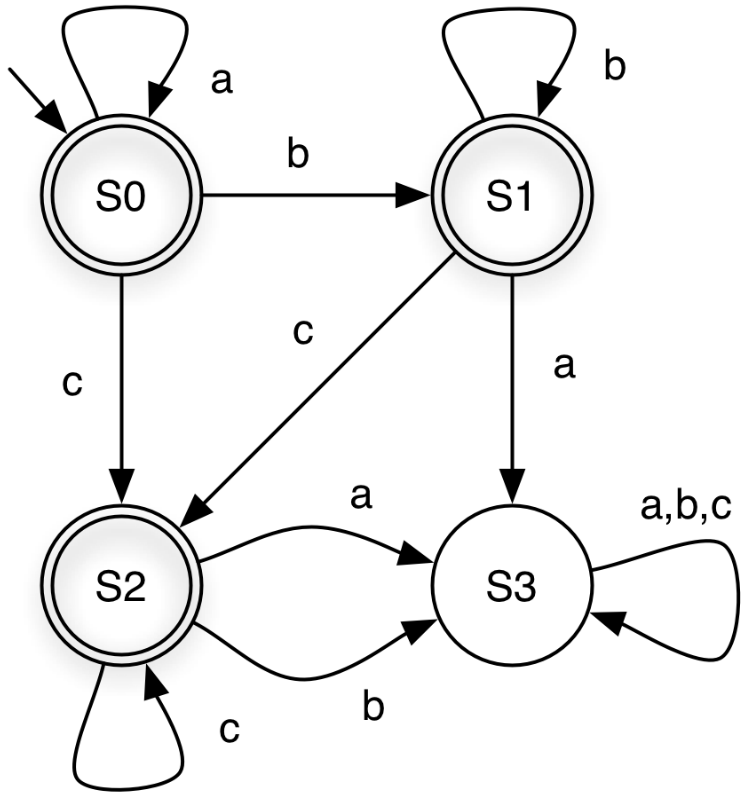
Finite Automaton: Example 3



string	state at end	accepts?
aabcc	S2	Y

(a,b,c notation shorthand for three self loops)

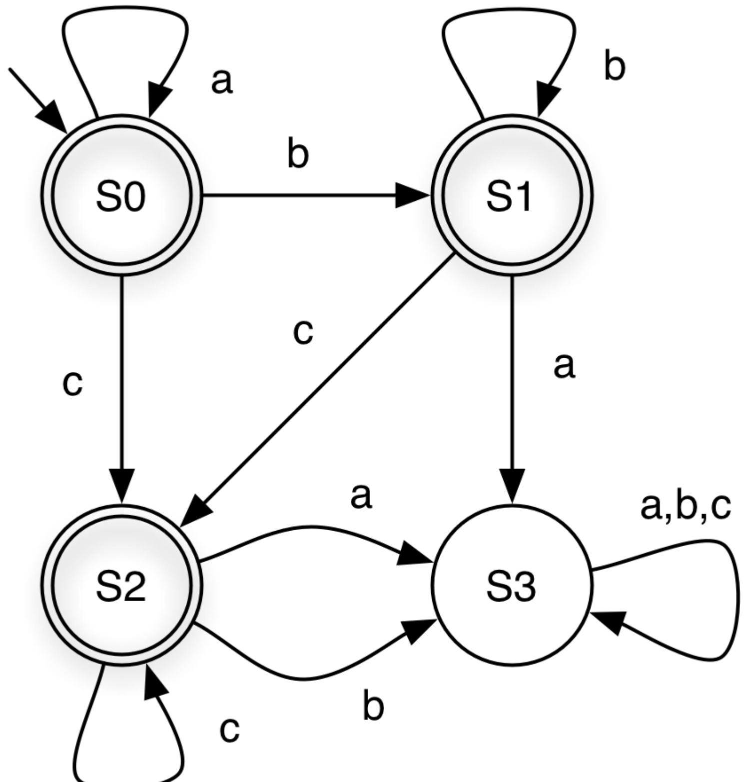
Finite Automaton: Example 3



string	state at end	accept s?
acca		

(a,b,c notation shorthand for three self loops)

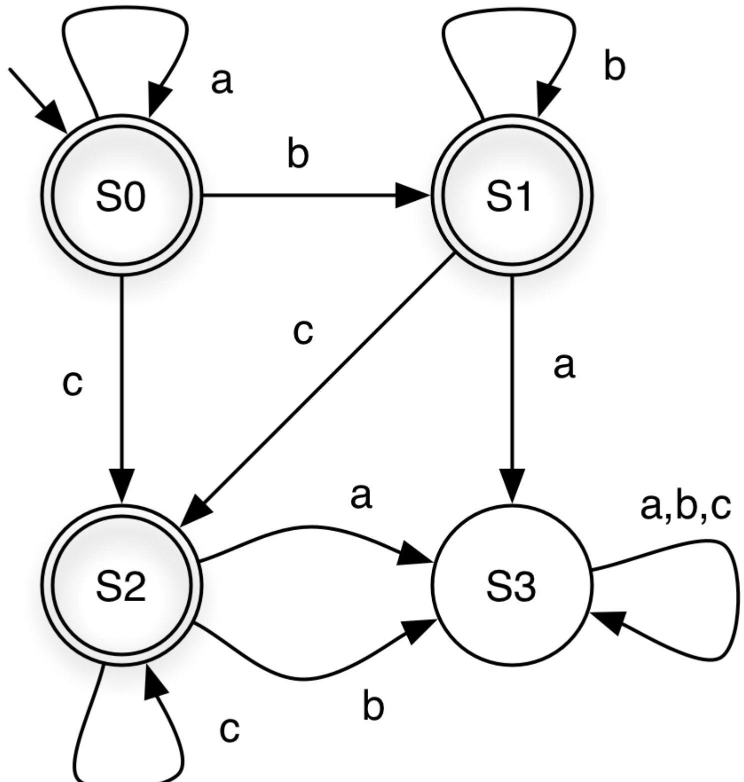
Finite Automaton: Example 3



string	state at end	accept s?
acca	S3	N

(a,b,c notation shorthand for three self loops)

Finite Automaton: Example 3



string	state at end	accept s?
aacbbb		

(a,b,c notation shorthand for three self loops)

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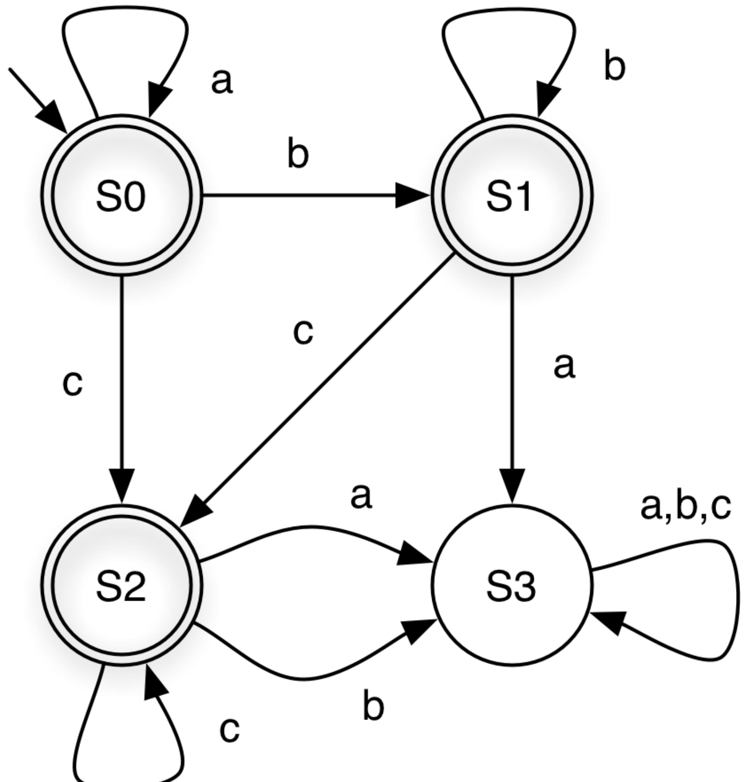

string	state at end	accept s?
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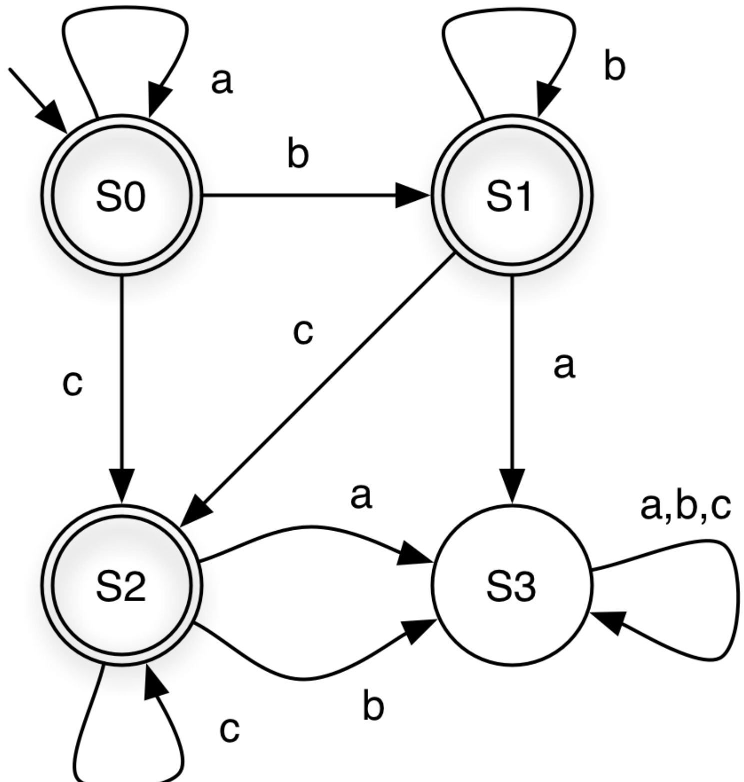
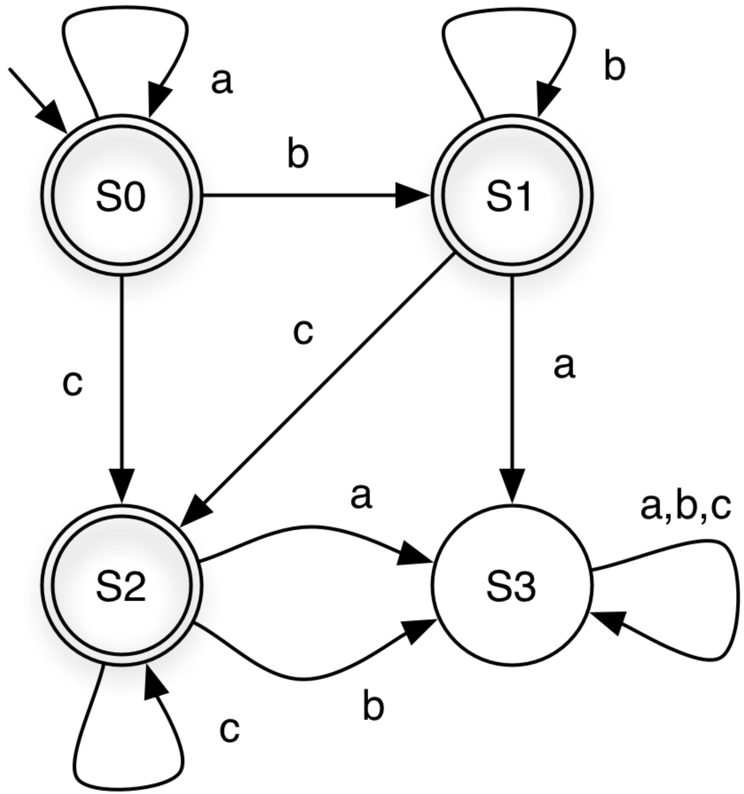


string	state at end	accept s?
ε		

(a,b,c notation shorthand for three self loops)

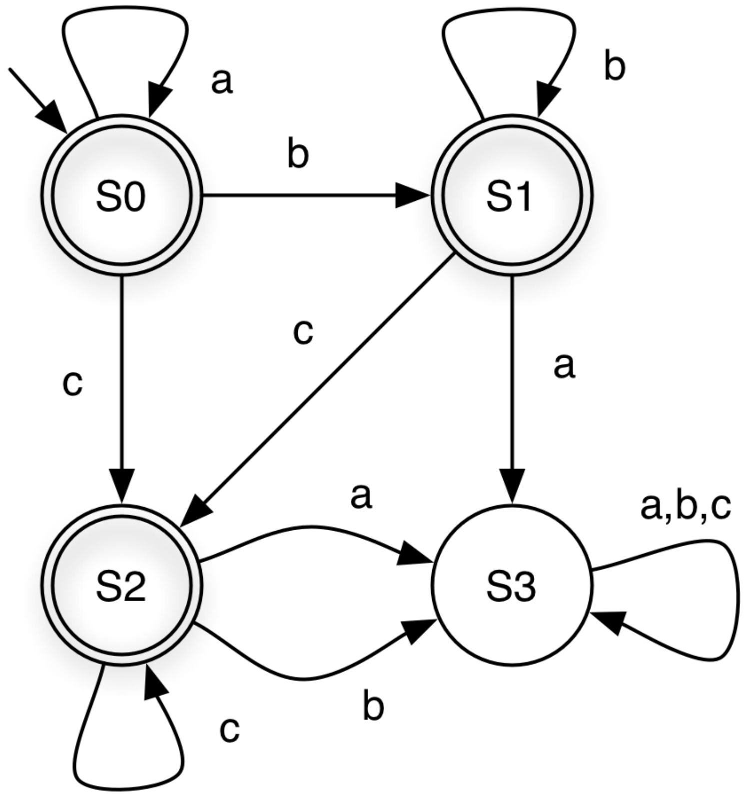
Finite Automaton: Example 3



string	state at end	accept s?
ε	S0	Y

(a,b,c notation shorthand for three self loops)

Finite Automaton: Example 3



string	state at end	accept s?
acba		

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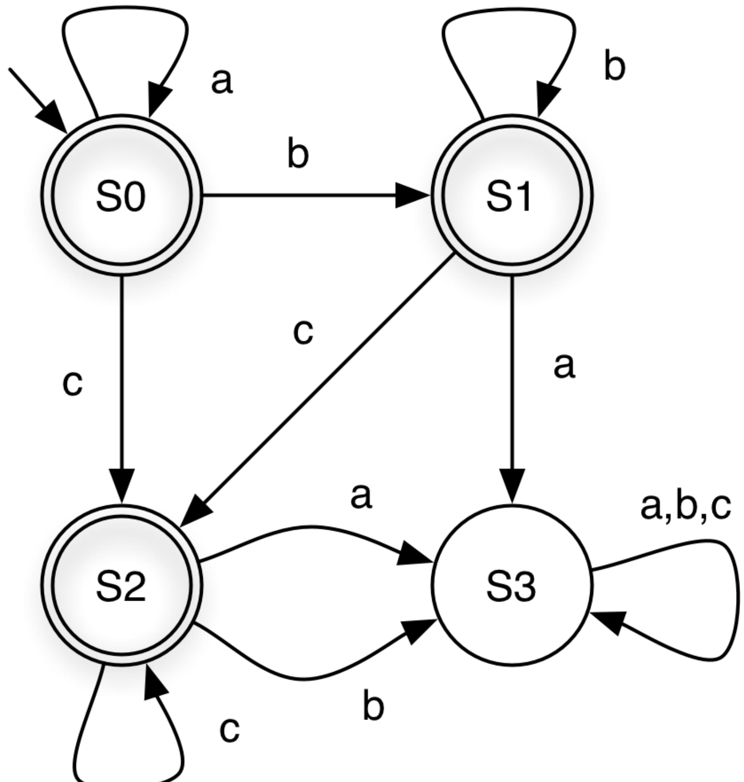
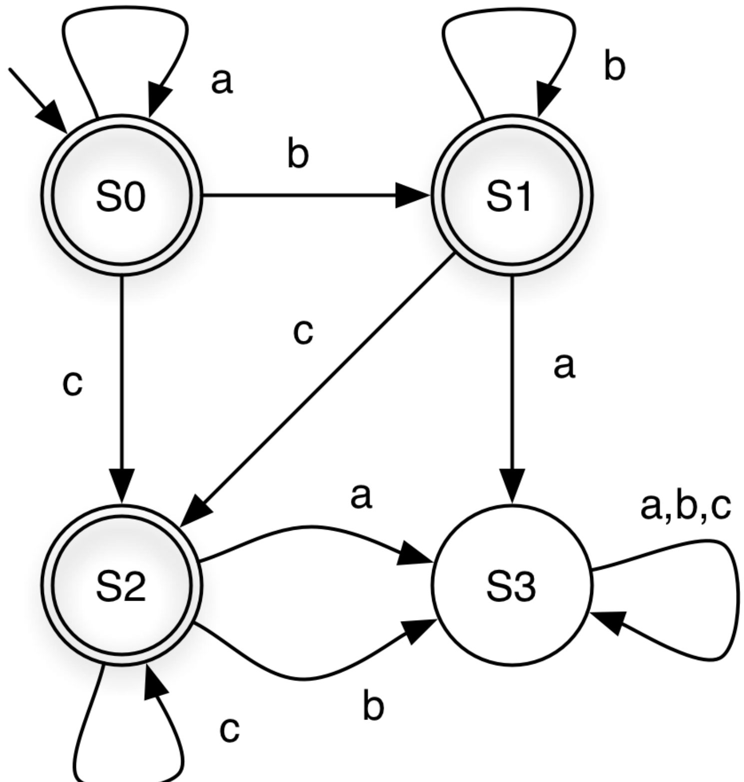
Finite Automaton: Example 3



string	state at end	accept s?
acba	S3	N

(a,b,c notation shorthand for three self loops)

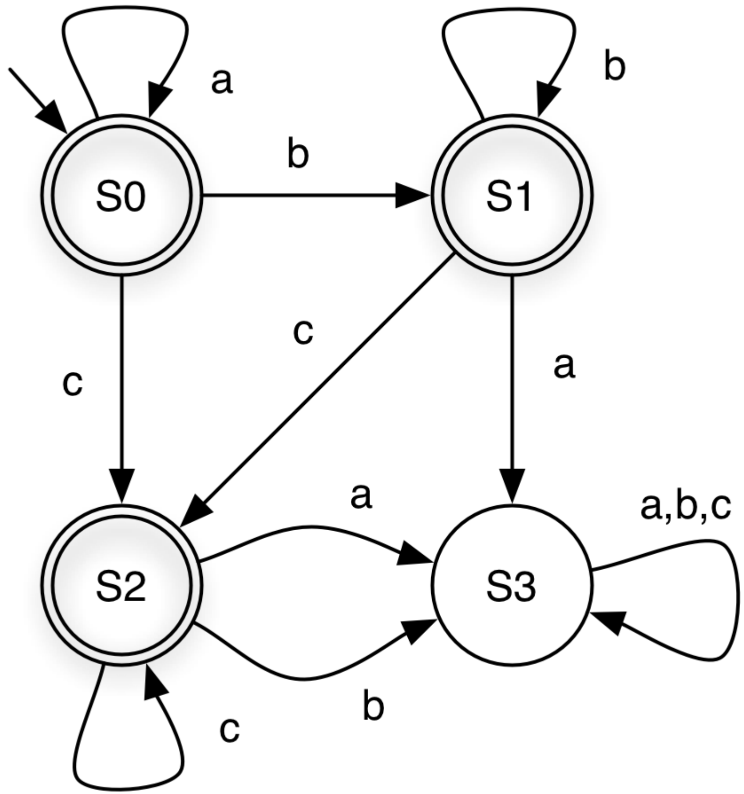
Quiz 4: Which string is **not** accepted?



A. bcca
B. abbbc
C. ccc
D. ε

(a,b,c notation shorthand for three self loops)

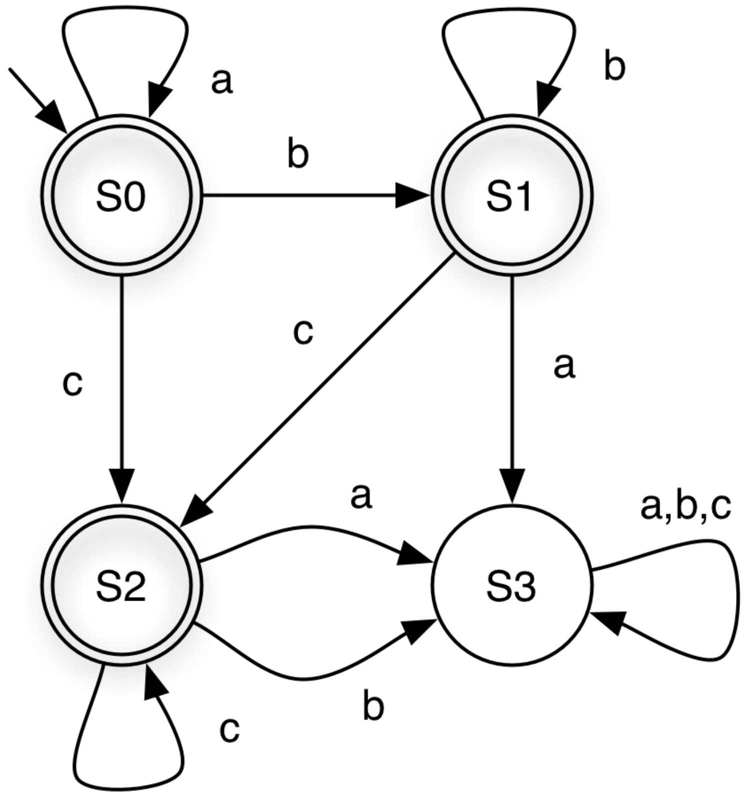
Quiz 4: Which string is **not** accepted?



A. bcca
B. abbbc
C. ccc
D. ε

(a,b,c notation shorthand for three self loops)

Finite Automaton: Example 3



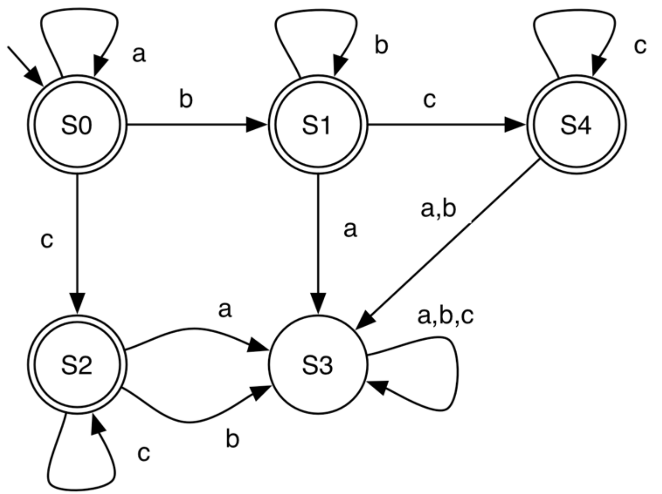
What language does this FA accept?

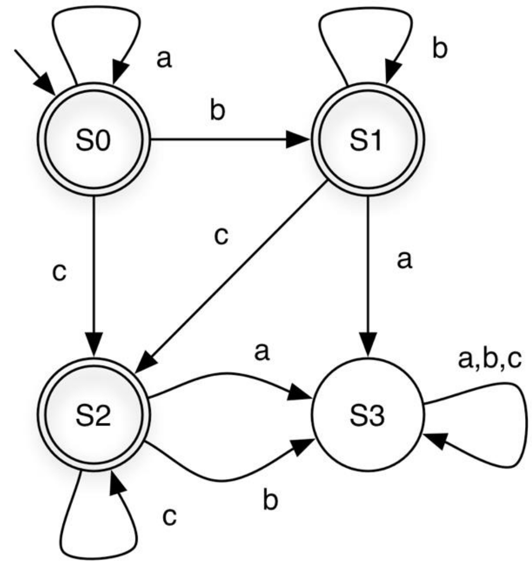
a*b*c*

S3 is a dead state – a nonfinal state with no transition to another state

- *aka* a trap state

Finite Automaton: Example 4



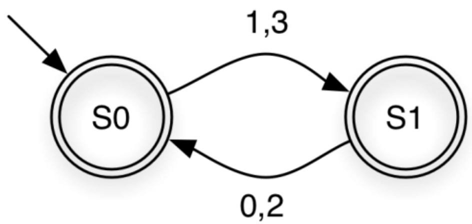


Language?

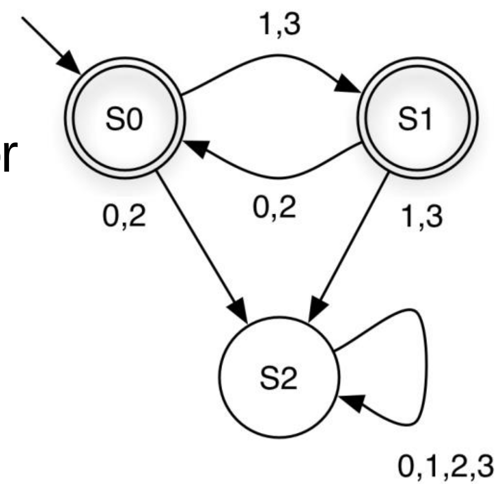
a*b*c* again, so FAs are not unique

Dead State: Shorthand Notation

- If a transition is omitted, assume it goes to a dead state that is not shown

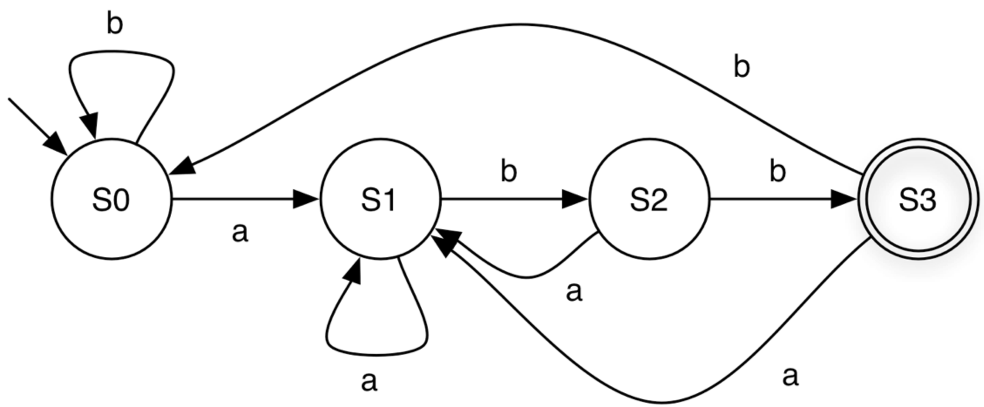


is short for



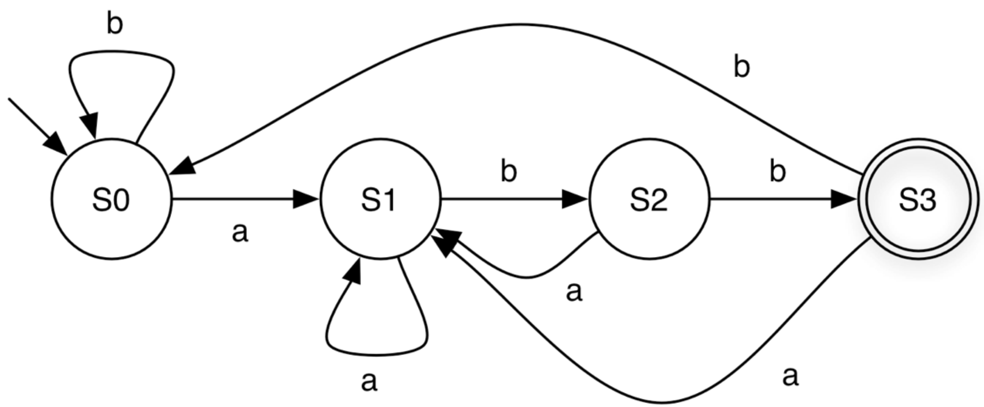
- Language?
  - Strings over {0,1,2,3} with alternating even and odd digits, beginning with odd digit

Finite Automaton: Example 5



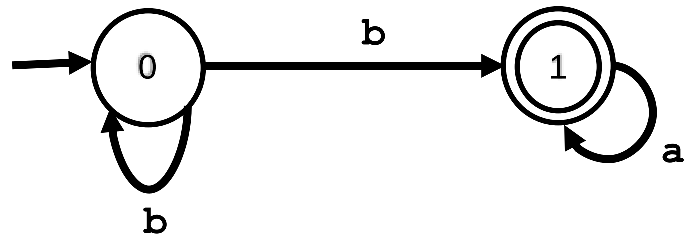
- Description for each state
  - S0 = "Haven't seen anything yet" *OR* "Last symbol seen was a b"
  - S1 = "Last symbol seen was an a"
  - S2 = "Last two symbols seen were ab"
  - S3 = "Last three symbols seen were abb"

Finite Automaton: Example 5



- Language as a regular expression?
  - (a|b)*abb

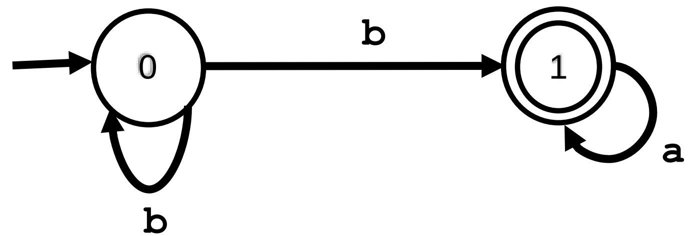
Quiz 5



Over Σ={a,b}, this FA accepts only:

A. A string that contains a single b.
B. Any string in {a,b}.
C. A string that starts with b followed by a’s.
D. One or more b’s, followed by zero or more a’s.

Quiz 5



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Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings containing two consecutive 0s followed by two consecutive 1s
- That accepts strings with an odd number of 1s
- That accepts strings containing an even number of 0s and any number of 1s
- That accepts strings containing an odd number of 0s and odd number of 1s
- That accepts strings that DO NOT contain odd number of 0s and an odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

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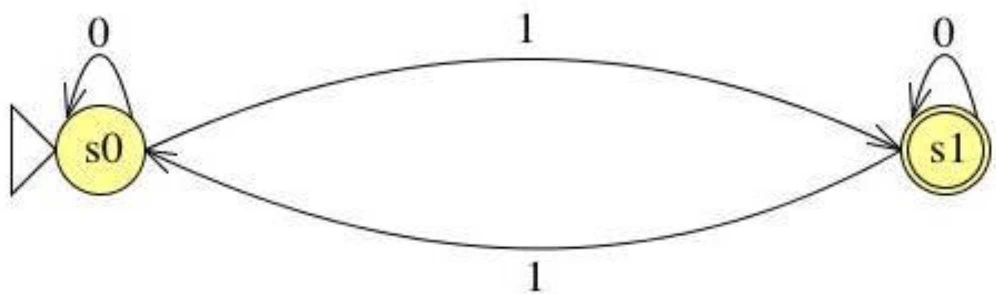

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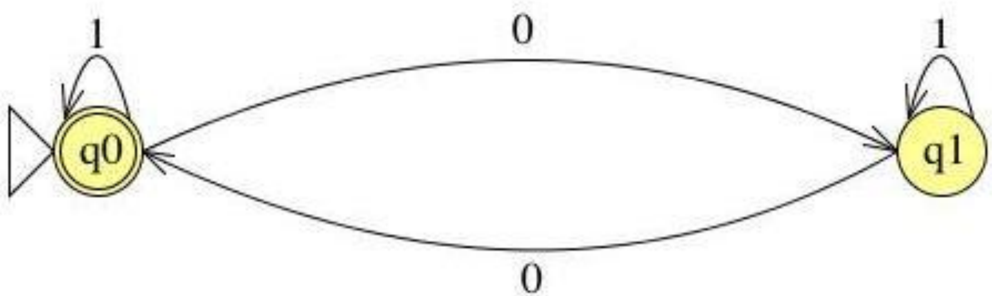
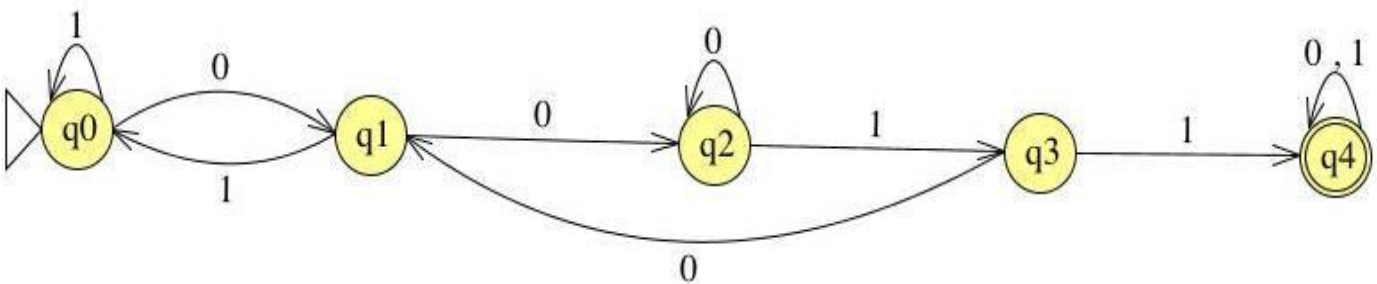


Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings containing two consecutive 0s followed by two consecutive 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings containing two consecutive 0s very immediately (right after, no other things in between) followed by two consecutive 1s

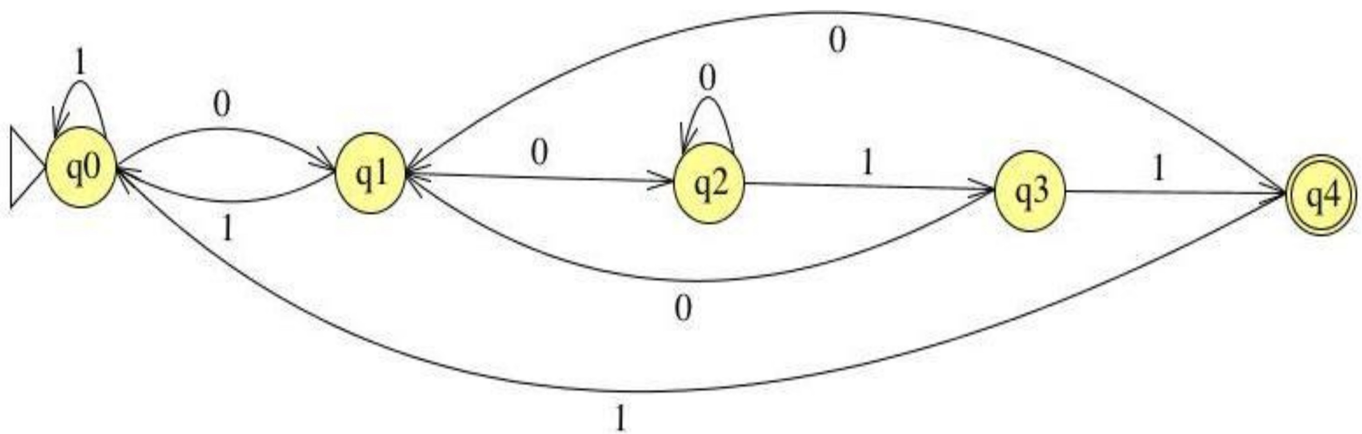


Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings end with two consecutive 0s followed by two consecutive 1s

Exercises: Define an FA over Σ = {0,1}

- That accepts strings end with two consecutive 0s followed by two consecutive 1s



Exercises: Define an FA over $\Sigma = \{0,1\}$

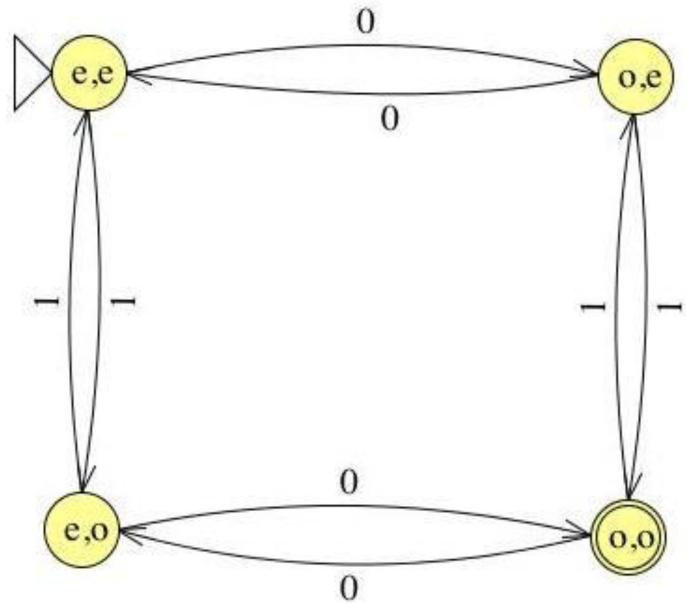
- That accepts strings containing an odd number of 0s and odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings containing an odd number of 0s and odd number of 1s

4 states:

0s	1s
e	e
o	e
e	o
o	o



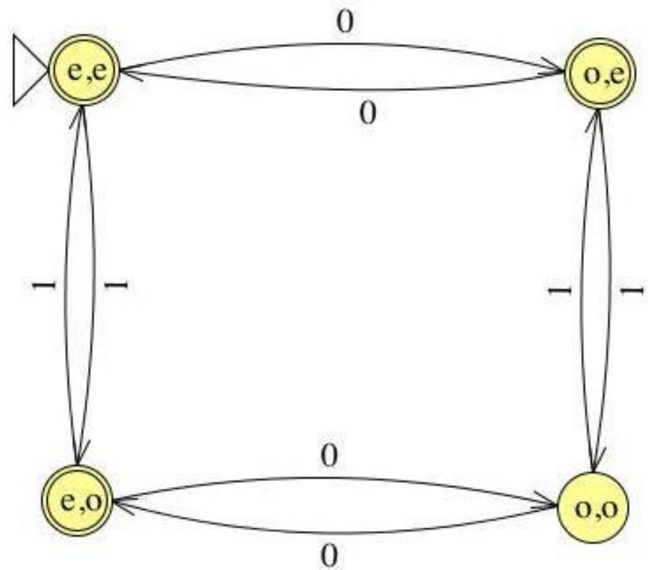
Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings that DO NOT contain odd number of 0s and an odd number of 1s

Exercises: Define an FA over Σ = {0,1}

- That accepts strings that DO NOT contain odd number of 0s and an odd number of 1s

Flip each state



Languages and Machines

A formal language is a set of strings of symbols drawn from a finite alphabet.

Can be specified either by

- a set of rules (such as regular expressions or a CFG) that generates the language
- a *formal machine* that *accepts* (*recognizes*) the language.