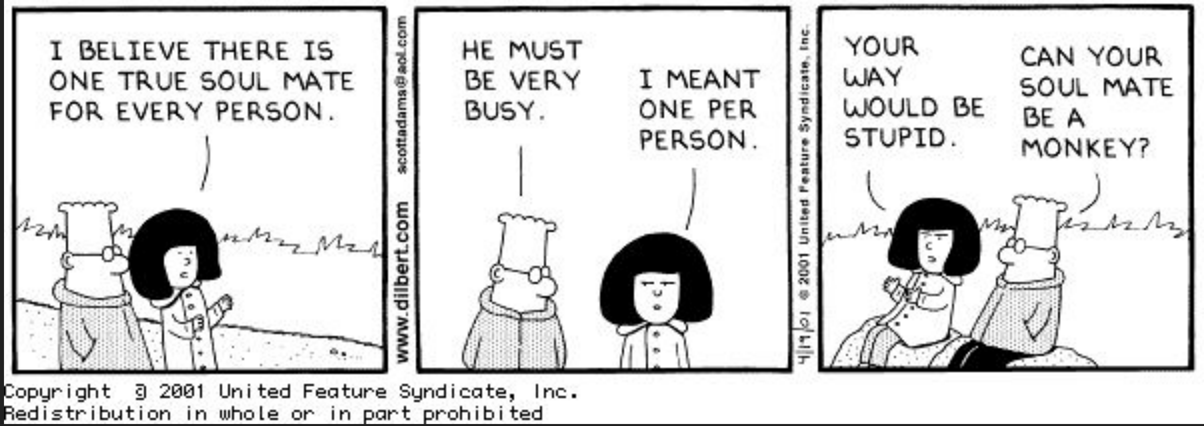
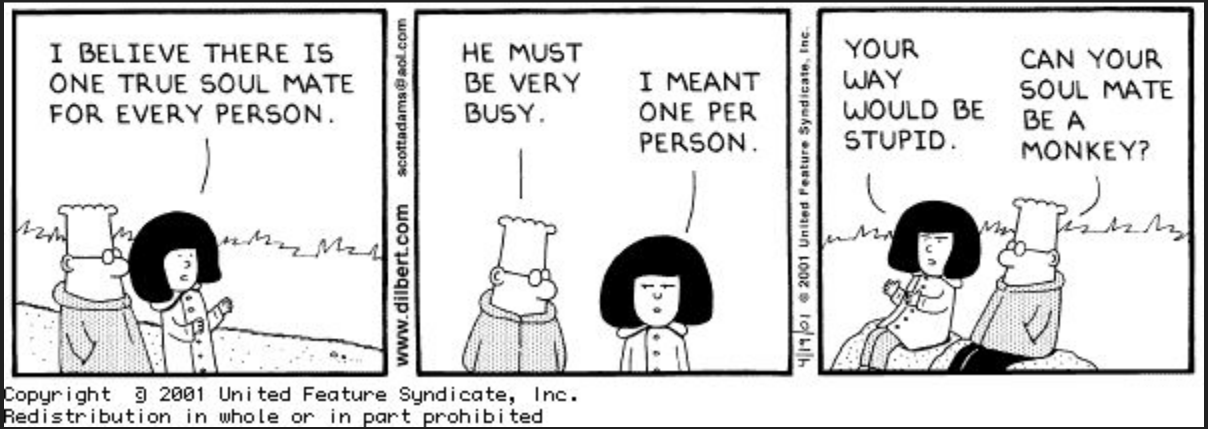
# Discrete Structures

Sets & Quantifiers
9/15/2022



# Quantifiers



# Quantifiers

Universal Quantifier

$\forall$ = “for every,” “for each,” “for any,” “given any,” or **“for all.”**

Existential Quantifier

$\exists$ = “there exists”

Uniqueness Quantifier (less common)

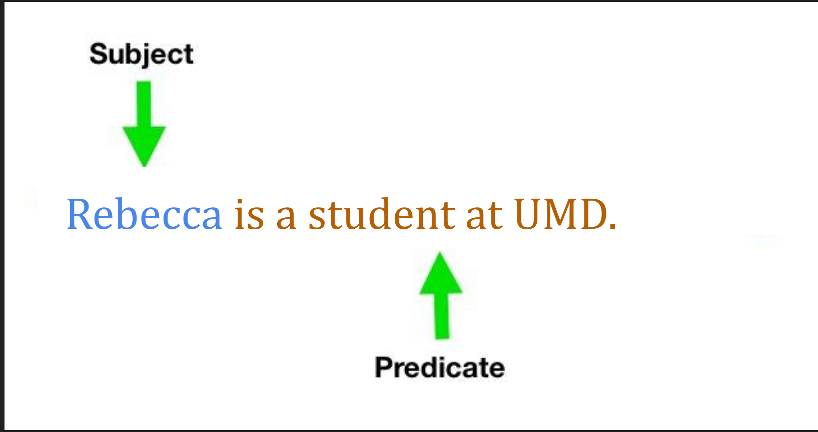
$\exists!$ = “there exists a unique”, “There is exactly one”

# Quantifiers

$\forall$ human beings x, x is mortal

# Predicate

In grammar, everything that isn't the subject



# Predicate

Rebecca is a student at UMD.

P(x) = *x* is a student at UMD.

GT(x,y) = x is greater than y.

# Predicate

sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

## Domain

set of all values that may be substituted in place of the variable.

# Predicate Domains

$P(x) = x$ is a student at UMD.

$GT(x,y) = x$ is greater than y.

# Predicate Domains

P(x) : x  is under 10 feet tall

P(me)

P(IRB)

P(my dog)

# Quantified Statement Examples

All vertebrates have spines.

$P(x) = x$ has a spine.
$\forall\ x \in \{\text{vertebrates}\}, P(x)$

$\forall\ x \in \{\text{vertebrates}\}, \text{HASSPINE}(x)$
$\forall\ x \in \{\text{vertebrates}\}$, x has a spine

You *may* see these written:

$(\forall\ x \in \{\text{vertebrates}\})[P(x)]$

# Quantified Statement Example

For every problem, there is a solution.

$\forall$ p $\in$ {problems}, $\exists$ s $\in$ {solutions} s.t. SOLVED(p,s)

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# Negating Quantifiers: ∀

¬(For every problem, there is a solution.)

≡

∃ at least one problem with no solution.

# Negating Quantifiers: $\forall$

$\neg(\forall\ p \in \{\text{problems}\}, \exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

$\equiv$

$\exists\ p \in \{\text{problems}\}, \neg(\exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

# Negating Quantifiers: ∀

All mathematicians wear glasses.

$\forall$ m $\in$ {mathematicians}, GLASSES(m)

$\neg$(All mathematicians wear glasses)

$\equiv$

There exists at least one mathematician who doesn't wear glasses.

# Negating Quantifiers: $\forall$

All mathematicians wear glasses.

$\forall$ m $\in$ {mathematicians}, GLASSES(m)

$\neg$($\forall$ m $\in$ {mathematicians}, GLASSES(m))

$\equiv$

$\exists$ m $\in$ {mathematicians}, $\neg$GLASSES(m)

# Negating Quantifiers: ∀

$\neg(\forall\ x \in D, Q(x))$

$\equiv$

$\exists\ x \in D, \neg Q(x)$

# Negating Quantifiers: ∃

$\neg(\exists\ x \in D, Q(x))$

$\equiv$

$\forall\ x \in D, \neg Q(x)$

# Negating Quantifiers: ∃

To prove:

“There does not exist a person taller than 10 feet”. $\neg(\exists\ x \in H, \text{GTTENFEETTALL}(x))$

You must prove:

“*Every* person is not taller than 10 feet” $\forall\ x \in H, \neg\text{GTTENFEETTALL}(x)$

# Negating Quantifiers: ∃

$\neg(\exists\ b \in \{\text{penguins}\}, \text{FLY}(b))$

$\equiv$

$\forall\ b \in \{\text{penguins}\}, \neg\text{FLY}(b)$

# Negating Quantifiers: $\forall$

$\neg(\forall\ p \in \{\text{problems}\}, \exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

$\equiv$

$\exists\ p \in \{\text{problems}\}, \neg(\exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

# Negating Quantifiers: ∀

¬ For every problem, there is a solution.

≡

$\neg(\forall\ p \in \{\text{problems}\}, \exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

≡

$\exists\ p \in \{\text{problems}\}, \neg(\exists\ s \in \{\text{solutions}\}\ \text{s.t. SOLVED}(p,s))$

≡

$\exists\ p \in \{\text{problems}\}, (\forall\ s \in \{\text{solutions}\}, \neg\text{SOLVED}(p,s))$

# CAREFUL with quantifiers

**∀, ∃** **vs** **∃, ∀**

"For all x in D, there exists y in F" vs "There exists y in F, s.t. for all x in D"

Example:

$(\forall\, x \in N, \exists\, y \in N)[x < y]$ vs $(\exists\, y \in N, \forall\, x \in N)[x < y]$

"For all nats, ∃ a nat greater" vs "∃ a nat s.t. It is greater than all others"

TRUE FALSE

# Sets

A set is a collection of elements.

- Order doesn't matter.
- Appearing multiple times in a set is same as once.
- *MEMBERSHIP* is what matters.

# Example Sets

$\{1, 2, 3\}$

$\{\text{USA, Canada, Mexico}\}$

{  }

$\{1, 2, 3, \ldots, 100\}$

# Set Notation

$x \in S$ means that x "is an element of" S

$x \notin S$ means that x "is NOT an element of" S

# Practice Sets

Let

$A = \{1, 2, 3\}$
$B = \{3, 1, 2\}$
$C = \{1, 1, 2, 3, 3, 3\}$

What are the elements of A, B, and C?

Is $\{0\} = 0$?

How many elements are in the set $\{1, \{1\}\}$?

For each nonnegative integer n, let $U_n = \{n, -n\}$. Find $U_1$, $U_2$, and $U_0$

$U_1=\{1,-1\}$, $U_2=\{-2,2\}$ $U_0= \{0, -0\} = \{0\}$

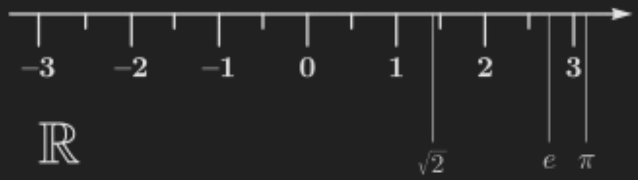
# Popular Sets

$\mathbb{N}$ Set of Natural Numbers. i.e. {0,1,2,3…}

$\mathbb{Z}$ Set of Integers. i.e. {...,-3,-2,-1,0,1,2,3…}

$\mathbb{Q}$ Set of all rational numbers i.e. $p/q$ s.t. $p,q \in Z \land q \neq 0$

$\mathbb{R}$ Set of Real Numbers .



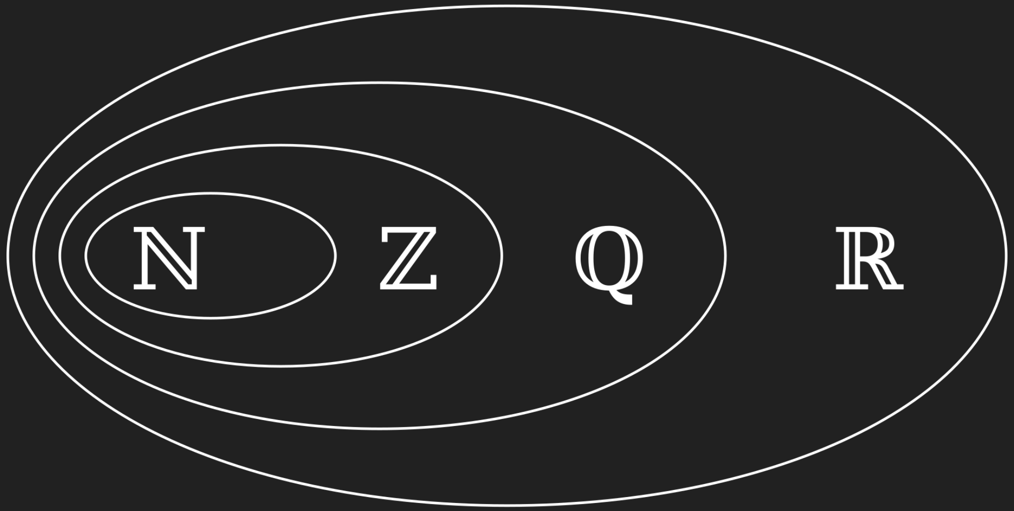

{} Empty set AKA “null set” $\emptyset$

# Elements in this set?

$\{ \emptyset, \{\} \}$

$\{ \emptyset, \{\emptyset\} \}$

# Popular Sets

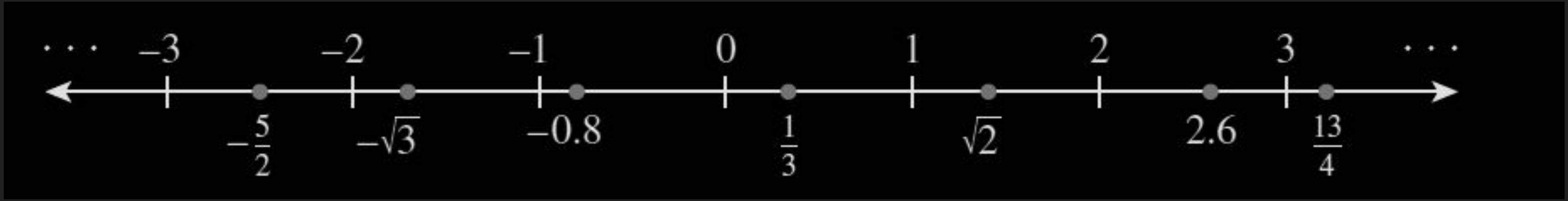


# Popular sets slightly modified

$R^+$

$Z^{nonneg}$

# Which set(s)?



# Set Builder Notation

$\{x \in S \mid P(x)\}$

$\{x \in R \mid -2 < x < 5\}$

$\{x \in Z \mid -2 < x < 5\}$

$\{x \in Z \mid -2 < x < 5\}$

# Set Builder Notation

$\{x \in S \mid P(x)\}$

$\{x \mid \; x \in R \land -2 < x < 5\}$

$\{x \mid x \in R \land -2 < x < 5\}$

$\{x \mid x \in R \land -2 < x < 5\}$

# Intervals

$[1,2] = \{x \in R \mid 1 \leq x \leq 2\}$

$[1,3) = \{x \in R \mid 1 \leq x < 3\}$

Note: $[1,2] \neq \{1,2\}$

# Practice Intervals

| Interval | Set |
|---|---|
| $(a,b)$ | |
| $[a,b)$ | |
| $[a, \infty)$ | |
| $(-\infty, b)$ | |
| $(-\infty, \infty)$ | |

# Practice Intervals

| Interval | Set |
|---|---|
| (a,b) | $\{x \in R \mid a < x < b\}$ |
| [a,b) | $\{x \in R \mid a \leq x < b\}$ |
| [a, ∞) | $\{x \in R \mid x \geq a\}$ |
| (-∞, b) | $\{x \in R \mid x < b\}$ |
| (-∞, ∞) | R |

# Subsets

A *is a subset of* B if all the elements in A are in B

$A \subseteq B$

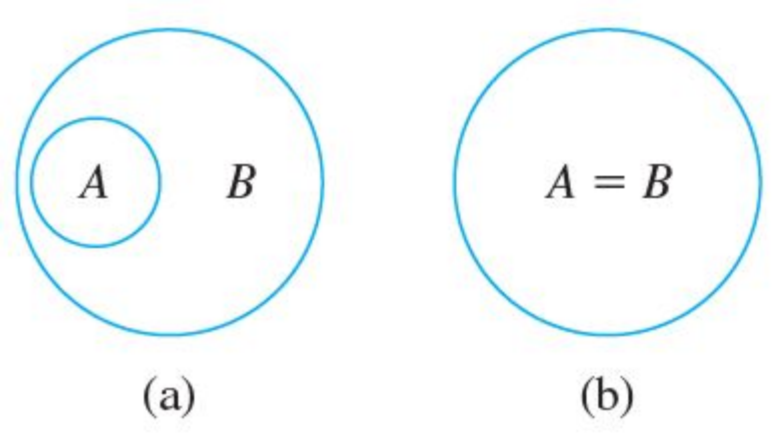
Examples:

$Z \subseteq Q$

$\{1,2,3\} \subseteq \{2,4,1,3,0\}$

$\{\} \subseteq \{2,4,1,3,0\}$

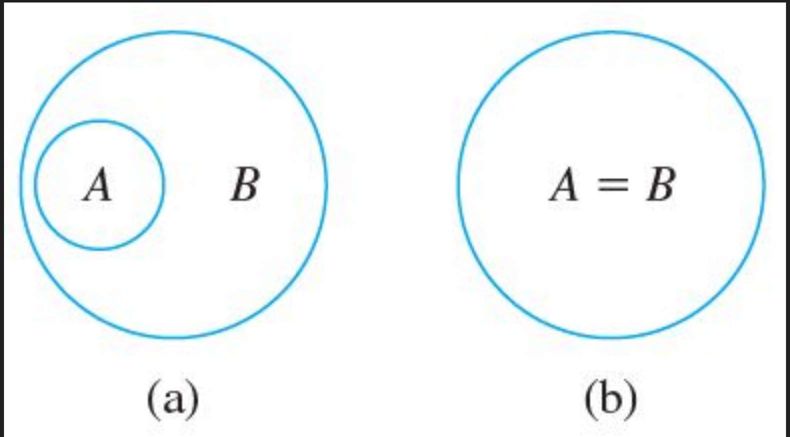
$\{1,2\} \subseteq \{2,1\}$



# Subsets

$A \subseteq B$

$A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$



# Subsets and Set Equality

$$A = B \Leftrightarrow ((A \subseteq B) \land (B \subseteq A))$$

$\{1,2,3\} \subseteq \{1,3,2\}$

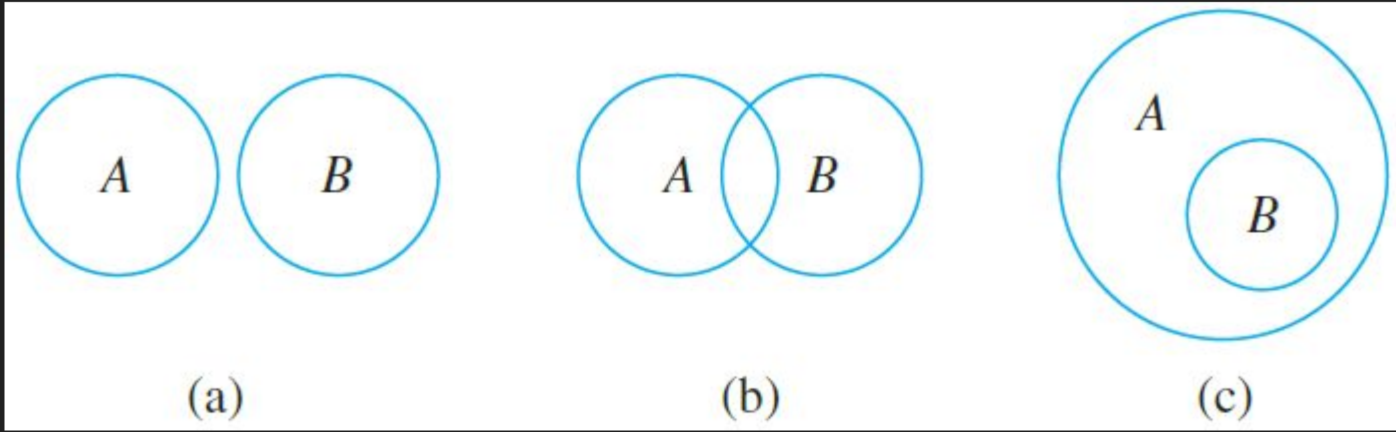
$\{1,3,2\} \subseteq \{1,2,3\}$

# Not a Subset

$A \not\subseteq B$ means: there is at least one element x such that $x \in A \land x \notin B$.

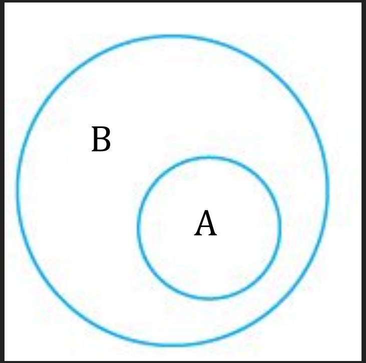
$\{1,2,3,4\} \not\subseteq \{1,2,3\}$

$\{a,b\} \not\subseteq \{a\}$



# Proper Subset

A is a proper subset of B *if, and only if,* every element of A is in B but there is at least one element of B that is not in A.




$A \subset B$

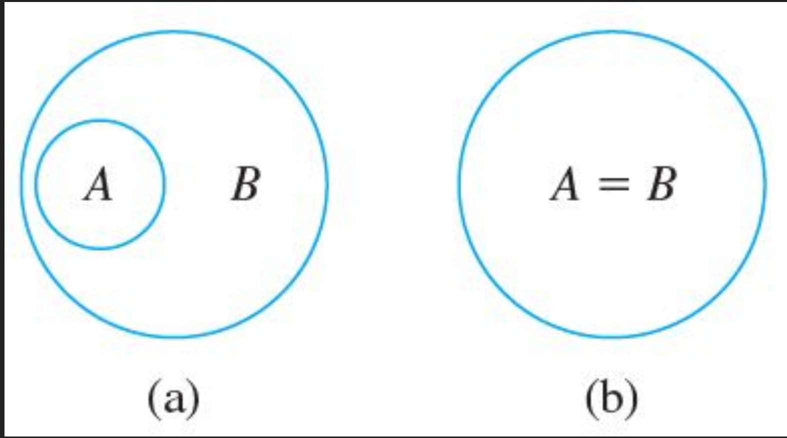
$A \subset B \Leftrightarrow ((x \in A \Rightarrow x \in B) \wedge (\exists\, y \in B, y \notin A))$

Example:
$Z \subset R$
$\{1,2\} \subset \{1,2,3\}$

# Proper Subset Analogy

$A \subset B \quad \approx \quad a < b$

$A \subseteq B \quad \approx \quad a \leq b$



# Practice

$A = Z^+$

$B = \{n \in Z \mid 0 \leq n \leq 100\}$

$C = \{100, 200, 300, 400, 500\}$

True or false?

1. $B \subseteq A$
2. $C \subset A$
3. $\exists$ x s.t. $x \in C \wedge x \in B$
4. $C \subseteq B$
5. $C \subseteq C$

# Practice

$A = \mathrm{Z}^{+}$

$B = \{\mathrm{n} \in \mathrm{Z} \mid 0 \leq \mathrm{n} \leq 100\}$

$C = \{100, 200, 300, 400, 500\}$

True or false?

1. $\mathrm{B} \subseteq \mathrm{A}$ FALSE
2. $\mathrm{C} \subset \mathrm{A}$ TRUE
3. $\exists$ x s.t. $x \in C \wedge x \in B$ TRUE
4. $\mathrm{C} \subseteq \mathrm{B}$ FALSE
5. $\mathrm{C} \subseteq \mathrm{C}$ TRUE

# Practice

True or false?

a. $2 \in \{1,2,3\}$
b. $\{2\} \in \{1,2,3\}$
c. $2 \subseteq \{1,2,3\}$
d. $\{2\} \subseteq \{1,2,3\}$
e. $\{2\} \subseteq \{\{1\},\{2\}\}$
f. $\{2\} \in \{\{1\},\{2\}\}$

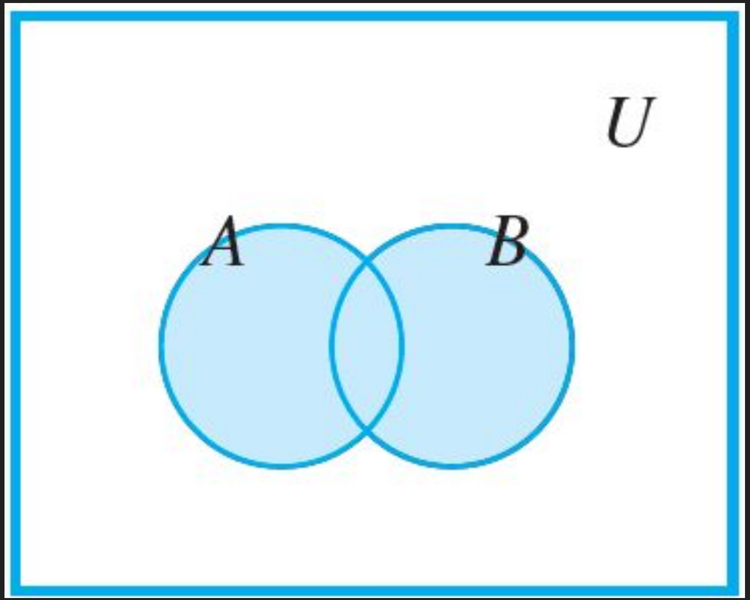
# Practice

True or false?

a. $2 \in \{1,2,3\}$ TRUE
b. $\{2\} \in \{1,2,3\}$
c. $2 \subseteq \{1,2,3\}$
d. $\{2\} \subseteq \{1,2,3\}$ TRUE
e. $\{2\} \subseteq \{\{1\},\{2\}\}$
f. $\{2\} \in \{\{1\},\{2\}\}$ TRUE

# Set Union

The union of A and B, denoted A ∪ B, is the set of all elements that are in at least one of A or B.



# Set Union

Examples:

$\{1,2,3\} \cup \{6,4\} =$

$\{1,2,3\} \cup \{1,2,3,4\} =$

$\{\} \cup \{6,4\} =$

# Set Union

Examples:

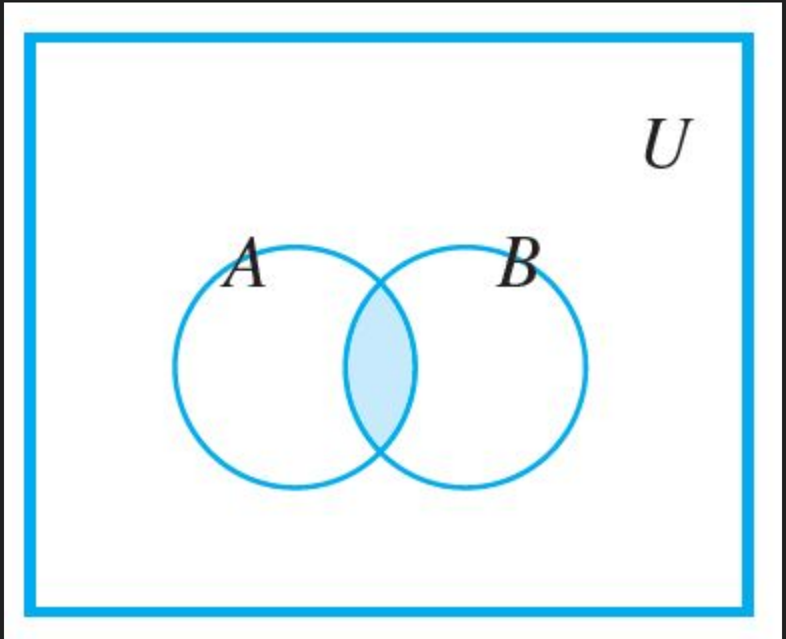
$\{1,2,3\} \cup \{6,4\} = \{1,2,3,4,6\}$

$\{1,2,3\} \cup \{1,2,3,4\} = \{1,2,3,4\}$

$\{\} \cup \{6,4\} = \{4,6\}$

# Set Intersection

The intersection of A and B, denoted A ∩ B, is the set of all elements that are common to both A and B.



# Set Intersection

Examples:

$\{1,2,3,4,5\} \cap \{4,5,6,7,8\} =$

$\{1,2,3\} \cap \{4,5\} =$

$\{1,2,3,4\} \cap \{\} =$

# Set Intersection

Examples:

$\{1,2,3,4,5\} \cap \{4,5,6,7,8\}=\{4,5\}$

$\{1,2,3\} \cap \{4,5\} = \{\}$

$\{1,2,3,4\} \cap \{\} = \{\}$

# Set Intersection

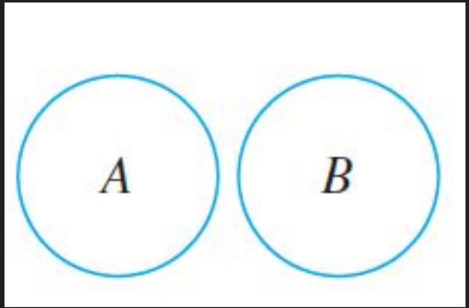
$\{1,2,3,4,5\} \cap \{4,5,6,7,8\} = \{4,5\}$

$\{1,2,3\} \cap \{4,5\} = \{\}$

$\{1,2,3,4\} \cap \{\} = \{\}$

Two sets are called **disjoint** *if, and only if,* they have no elements in common.

A and B are disjoint $\Leftrightarrow A \cap B = \{\}$

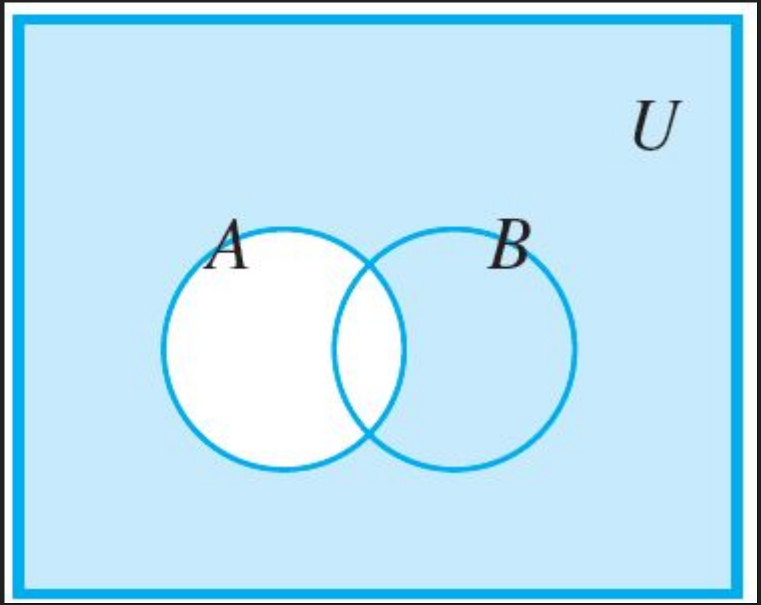


# Set Complement

Context = *universal set* U

The complement of A, denoted $A^c$ ,or $A'$ is the set of all elements in U that are not in A.

$A' = \{ x \in U \mid x \notin A\}$




Example:

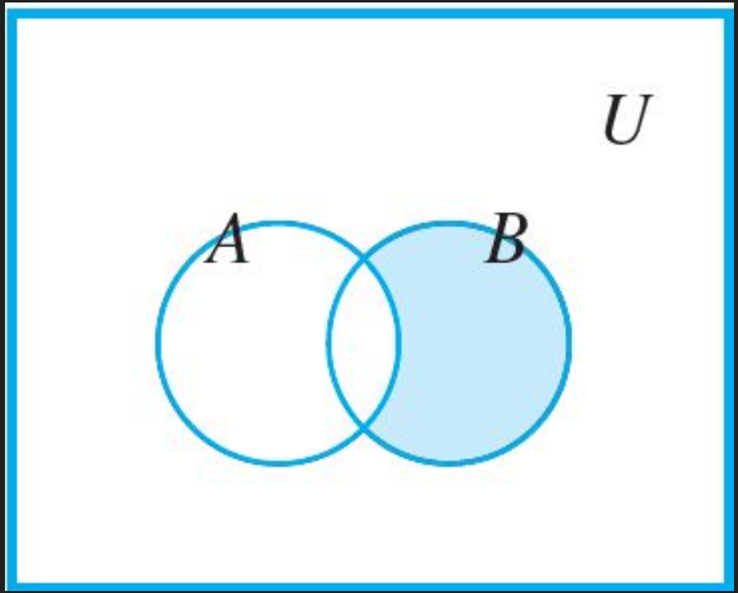
U=Z: $\{\text{evens}\}^c = \{\text{odds}\}$

# Set Difference

The difference of B minus A (or relative complement of A in B), denoted B - A, is the set of all elements that are in B and not A.

$B - A = \{x \mid (x \in B) \wedge (x \notin A)\}$

Sometimes also written: B/A



# Set Difference

Examples:

$\{1,2,3,4\}-\{3,4\} =$

$\{1,2,3,4\}-\{3,4,5,6\}=$

$\{3,4,5,6\}-\{1,2,3,4\}=$

$\{a,b,c\} - \{\} =$

# Set Difference

Examples:

$\{1,2,3,4\} - \{3,4\} = \{1,2\}$

$\{1,2,3,4\} - \{3,4,5,6\} = \{1,2\}$

$\{3,4,5,6\} - \{1,2,3,4\} = \{5,6\}$

$\{a,b,c\} - \{\} = \{a,b,c\}$

# More Practice

$U = \{a,b,c,d,e,f,g\}$
$A = \{a,c,e,g\}$
$B = \{d,e,f,g\}$

Find:

$A \cup B =$

$A \cap B =$

$B - A =$

$A^c =$

# More Practice

$U = \{a,b,c,d,e,f,g\}$

$A = \{a,c,e,g\}$

$B = \{d,e,f,g\}$

Find:

$A \cup B = \{a,c,d,e,f,g\}$

$A \cap B = \{e,g\}$

$B - A = \{d,f\}$

$A^c = \{b,d,f\}$

# Critical Thinking

$$(A \cup B = A) \Rightarrow (A = B)$$

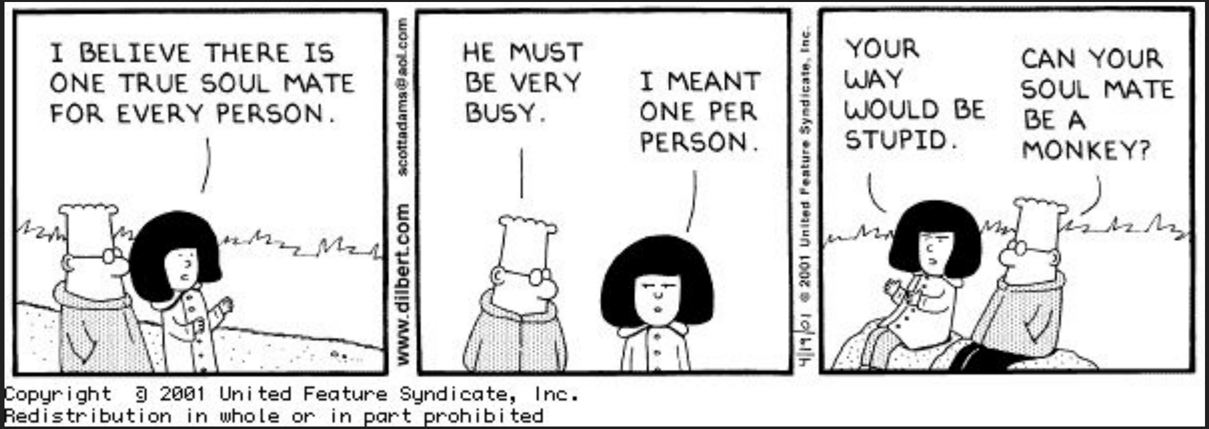
$$(A \cap B = A) \Rightarrow (A = B)$$

# Summary

- Quantifiers
  - ∀
  - ∃
- Sets
  - Notations
    - List elements
    - Common sets
    - Common sets modified
    - Set builder
  - Subset
  - Equality
  - Proper subset
  - Union
  - Intersection
  - subtraction/difference
    - Disjoint sets

# Quantifiers

# Quantifiers




$\forall$ x $\in$ {humans}, $\exists!$ m $\in$ {humans} s.t.
SOULMATE(x,m) $\wedge$
$\forall$ z $\in$ {humans} – {x}, $\neg$ SOULMATE(z,m)